Preface

The $k(GV)$ problem has recently been solved, completing the work of a series of authors over a period of more than forty years. The objective of this book is to describe the developments, the ideas and methods, leading to this remarkable result. All details of the proof will be presented.

Let G be a finite group. The number $k(G)$ of conjugacy classes of G, which is just the number of irreducible complex characters of G, is certainly an invariant of G of special interest. For some families of finite groups, like the symmetric groups and some of the finite classical groups, this invariant is known to some extent. From the point of view of abstract group theory, however, little can be said about $k(G)$ in relation to the group G. Of course $k(G) \leq |G|$, with equality if and only if G is abelian. If N is a normal subgroup of G, then it is easy to see that $k(G) \leq k(N) \cdot k(G/N)$ but there are examples where $k(N) > k(G)$ (e.g. in the dihedral group of order 10). This makes it difficult to bound $k(G)$ by inductive methods.

A unifying notion in group theory is the concept of representation. So finite groups often appear as subgroups of permutation groups or linear groups. The "geometry" of a group (as permutation group or linear group, or as a group of Lie type *etc.*) should be used in order to describe basic invariants. Here we just assume that G is embedded into the linear group $\mathrm{GL}(V)$ of some finite vector space V over some prime field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, and we wish to bound $k(G)$ by a function of $|V| = p^m$. A weak form of the $k(GV)$ problem is the question whether $k(G) \leq |V| - 1$ in the case where G is a p'-group ($p \nmid |G|$). Here equality can happen since $\mathrm{GL}(V)$ contains cyclic subgroups of order $|V|-1$, the so-called Singer cycles. The restriction to p'-groups is essential, because $\mathrm{GL}(V) = \mathrm{GL}_m(p)$ has abelian subgroups of order $p^{\lfloor \frac{m^2}{4} \rfloor}$.

Now consider the semidirect product GV (also written $G \ltimes V$ or $V : G$). Since $k(GV) > k(G)$, one can strengthen the question above by asking whether $k(GV) \leq |V|$ when G is a p'-subgroup of $\mathrm{GL}(V)$. This is known as the $k(GV)$ conjecture.

The $k(GV)$ conjecture is a special case of Brauer's celebrated $k(B)$ problem [Brauer, 1956]. The $k(B)$ conjecture predicts that the number $k(B)$ of ordinary irreducible characters in a p-block B is bounded above by the order of a defect group D of B (Problem (X) in [Feit, 1982]). In the situation of the $k(GV)$ conjecture, the group GV has a single p-block with the unique defect group $D = V$. [Brauer and Feit, 1959] established the general upper bound $k(B) \leq 1 + \frac{1}{4}|D|^2$. This bound has since only been slightly improved. The conjectured estimate $k(B) \leq |D|$ has been proved for cyclic and rank 2 abelian defect groups D, and for some families of finite groups, including the symmetric and finite general linear groups. There is reason to believe that the $k(B)$ conjecture is one of the most difficult problems proposed by Brauer.

[Nagao, 1962] noticed that the $k(B)$ conjecture holds for p-solvable groups provided it holds for semidirect products as discussed above. So the $k(GV)$ theorem implies Brauer's conjecture for such groups. This was the original motivation for treating the $k(GV)$ problem which is, on the other hand, of interest in its own right.

In the proof of the $k(GV)$ conjecture Clifford theory plays an important role. This theory may be loosely described as using normal subgroups to pass from the module of interest to lower-dimensional, more tractable modules. This can be fruitfully applied to many problems in group representation theory. In a typical sequence of reductions one tries to show that the module in question may be assumed to be irreducible, then absolutely irreducible, then primitive, then tensor indecomposable, and finally tensor primitive. If one tries this approach on the $k(GV)$ problem, where V is a faithful coprime $\mathbb{F}_pG$-module, the first two steps are easy, but the third is not. Indeed, the final stage of the proof of the $k(GV)$ conjecture was the rather difficult verification of the problem when V is induced from a certain module with cardinality 5^2.

Nevertheless, Clifford theory is crucial, in a manner we now wish to explain. We consider the centralizers (stabilizers) $C_G(v)$ as v ranges over the vectors in V. It is easy to see that $k(GV)$ is the sum over the $k(C_G(v))$ when v ranges over a set of representatives for the G-orbits on V. Remarkably, however, the main theoretical results in the solution of the $k(GV)$ problem assert that the existence of *one* vector v in V with $C_G(v)$ satisfying suitable conditions is enough to imply that $k(GV) \leq |V|$.

The most elementary of these centralizer criteria asserts that it suffices to find $v \in V$ such that $C_G(v) = 1$ (Theorem 1.5d of this book). Knörr established two more general centralizer criteria which had great influence on later work [Knörr, 1984]. He showed first that $k(GV) \leq |V|$ if $C_G(v)$

is abelian for some v in V (Theorem 3.4d). Even more important was his second criterion, because it allowed one to assume, in many cases, that V is primitive, thus partially overcoming the major obstacle to the direct Clifford-theoretic approach. Knörr's ideas are related to some general techniques developed in [Brauer, 1968].

Gow advantageously reworked Knörr's ideas and proved the result in the case that V is a self-dual G-module [Gow, 1993]. In [Robinson, 1995] it was noticed that it suffices to find $v \in V$ such that the restriction of V to $C_G(v)$ is self-dual. The most powerful criterion then was established in [Robinson and Thompson, 1996]. The Robinson–Thompson criterion (Theorem 5.2b) asserts that $k(GV) \leq |V|$ if there exists v in V such that the restriction of V to $C_G(v)$ contains a faithful self-dual summand (with real-valued Brauer character). Such a vector v will be called a *real vector.*

This criterion is sufficiently easy to verify, and it is compatible with the Clifford-theoretic reduction, and so led to much progress towards a solution of the $k(GV)$ problem. At the end of the reduction steps one is left with a pair (G, V) admitting no real vectors, such that the generalized Fitting subgroup of G has the form $E \cdot Z(G)$, where E is normal in G and absolutely irreducible on V, and where either E is a group of extraspecial type or is quasisimple. Such a pair (G, V) will be called *nonreal reduced.* Robinson and Thompson already showed that this can happen only when the characteristic $p \leq 5^{30}$.

In Chapters 6 and 7, we use counting arguments to show that nonreal reduced pairs do exist only when p is $3, 5, 7, 11, 13, 19$ or 31 and V has dimension $2, 3$ or 4. It follows that the $k(GV)$ conjecture is true when the characteristic p is not one of these seven primes. We give a new treatment of the extraspecial case, using ideas from [Robinson, 1997], [Riese–Schmid, 2000], [Gluck–Magaard, 2002a] and [Riese, 2002]. In the quasisimple case our approach is based on work of [Liebeck, 1996], [Goodwin, 2000], [Riese, 2001] and [Köhler–Pahlings, 2001]. Much is simplified by systematically using properties of characters related to extraspecial groups (Heisenberg groups). The counting techniques usually lead to "small" groups G and modules V of "small" degrees, often affording minimal "Weil characters".

After the analysis of the extraspecial and quasisimple cases was completed around 2000, one had to deal with arbitrary pairs (G, V) admitting no real vectors. In some Clifford-theoretic sense then (G, V) "involves" a nonreal reduced pair, but *a priori* this seemed to be hard to control. So it was striking, and a surprise, that V is just obtained by module induction from a nonreal reduced pair, and that G is close to being a full wreath product with respect to the corresponding imprimitivity decomposition of V (Theorems 8.4 and 8.5c). This crucial result was accomplished in [Riese–Schmid, 2003] using ideas developed in [Gluck–Magaard, 2002b].

In the final stages of the proof of the $k(GV)$ theorem, one therefore had to treat imprimitive modules induced from nonreal reduced modules. Here one requires bounds on the number of conjugacy classes in a permutation group. Liebeck and Pyber have established the general upper bound 2^{n-1} for the number of conjugacy classes in a permutation group of degree n (Theorem 9.3). In this manner the proof was completed by Riese and Schmid for $p \neq 5$, and by Gluck, Magaard, Riese and Schmid for $p = 5$.

Chapter 11 addresses the question of when one can have equality $k(GV) = |V|$, without giving a conclusive answer, however. In the final two chapters we briefly describe some consequences of the $k(GV)$ theorem for general block theory and consider the problem of bounding $k(GV)$ when G is completely reducible on V but $|G|$ and $|V|$ are not coprime.

At present the proof of the $k(GV)$ conjecture relies on the classification of the finite simple groups; see Chapters 7 and 9. But often I argue on the basis of general counting arguments, which do not refer to a certain simple group and which reduce the discussion to groups of low order. Most challenging are indeed the classical groups. For solvable groups the proof is independent of the classification theorem.

I hope this monograph will be comprehensible to a graduate student with some background in group theory and representation theory. Much of this general background is provided by Isaacs' book [Isaacs, 1976]. Some knowledge of the finite simple groups is also needed; the "Atlas of Finite Groups" [Conway *et al.*, 1985] contains much of the necessary information. For the convenience of the reader appendices are included on the cohomology of finite groups, on parabolic subgroups of some finite classical groups, and on the Weil characters of such groups.

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