Chapter 1

Galileo

1.1 Principle of Galilean relativity

Galileo Galilei

Principles of relativity address the problem of how events that occur in one place or state of motion are observed from another. And if events occurring in one place or state of motion look different from those in another, how should one determine the laws of motion?
Galileo approached this problem via a thought experiment which imagined observations of motion made inside a ship by people who could not see outside.

Galileo showed that the people isolated inside a uniformly moving ship are *unable to determine by measurements made inside it whether they are moving!*

> ... have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that. You will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still.
> – Galileo Galilei, *Dialogue Concerning the Two Chief World Systems* [Ga1632]

Galileo's thought experiment showed that a man who is below decks on a ship cannot tell whether the ship is docked or is moving uniformly through the water at constant velocity. He may observe water dripping from a bottle, fish swimming in a tank, butterflies flying, etc. Their behaviour will be just the same, whether the ship is moving or not.

Definition 1.1.1 (Galilean transformations)
*Transformations of reference location, time, orientation or state of uniform translation at constant velocity are called **Galilean transformations**.*

Definition 1.1.2 (Uniform rectilinear motion)
*Coordinate systems related by Galilean transformations are said to be in **uniform rectilinear motion** relative to each other.*

Galileo's thought experiment led him to the following principle.

Definition 1.1.3 (Principle of Galilean relativity)
The laws of motion are independent of reference location, time, orientation, or state of uniform translation at constant velocity. Hence, these laws are invariant under Galilean transformations.

Remark 1.1.4 (Two tenets of Galilean relativity)
Galilean relativity sets out two important tenets:
(1) It is impossible to determine who is actually at rest; and
(2) Objects continue in uniform motion unless acted upon.
*The second tenet is known as **Galileo's Law of Inertia**.*
*It is also the basis for **Newton's First Law of Motion**.*

1.2 Galilean transformations

Definition 1.2.1 (Galilean transformations)
Galilean transformations of a coordinate frame consist of space-time translations, rotations and reflections of spatial coordinates, as well as Galilean "boosts" into uniform rectilinear motion.

In three dimensions, the Galilean transformations depend smoothly on ten real parameters, as follows:

- **Space-time translations,**

$$g_1(\mathbf{r}, t) = (\mathbf{r} + \mathbf{r}_0, t + t_0)\,.$$

These possess four real parameters: $(\mathbf{r}_0, t_0) \in \mathbb{R}^3 \times \mathbb{R}$, *for the three dimensions of space, plus time.*

- **Spatial rotations and reflections,**

$$g_2(\mathbf{r}, t) = (O\mathbf{r}, t)\,,$$

for any linear orthogonal transformation $O : \mathbb{R}^{3N} \mapsto \mathbb{R}^{3N}$ *with* $O^T = O^{-1}$. *These have three real parameters, for the three axes of rotation and reflection.*

- **Galilean boosts** *into uniform rectilinear motion,*

$$g_3(\mathbf{r}, t) = (\mathbf{r} + \mathbf{v}_0 t, t)\,.$$

These have three real parameters: $\mathbf{v}_0 \in \mathbb{R}^3$, *for the three directions and rates of motion.*

Definition 1.2.2 (Group)
*A **group** G is a set of elements that possesses a binary product (multiplication), $G \times G \to G$, such that the following properties hold:*

1. *The product gh of g and h is associative, that is, $(gh)k = g(hk)$.*
2. *An identity element exists, e: $eg = g$ and $ge = g$, for all $g \in G$.*
3. *Inverse operation $G \to G$, so that $gg^{-1} = g^{-1}g = e$.*

Definition 1.2.3 (Lie group)
*A **Lie group** is a group that depends smoothly on a set of parameters. That is, a Lie group is both a group and a smooth manifold, for which the group operations are smooth functions.*

Proposition 1.2.4 (Lie group property)
Except for reflections, Galilean transformations form a Lie group.

Proof. Any Galilean transformation $g \in G(3) : \mathbb{R}^{3N} \times \mathbb{R} \mapsto \mathbb{R}^{3N} \times \mathbb{R}$ may be expressed uniquely as a composition of the three basic transformations $\{g_1, g_2, g_3\} \in G(3)$. Consequently, the set of elements comprising the transformations $\{g_1, g_2, g_3\} \in G(3)$ closes under the binary operation of composition. The Galilean transformations also possess an identity element $e : eg_i = g_i = g_i e$, $i = 1, 2, 3$, and each element g possesses a unique inverse g^{-1}, so that $gg^{-1} = e = g^{-1}g$. These are the defining relations of a group. The smooth dependence of the group of Galilean transformations on its ten parameters means the ***the Galilean group*** $G(3)$ ***is a Lie group*** (except for the reflections, which are discrete, not smooth). ■

Remark 1.2.5
Compositions of Galilean boosts and translations commute. That is,

$$g_1 g_3 = g_3 g_1 \,.$$

However, the order of composition does matter in the composition of Galilean transformations when rotations and reflections are involved. For example, the action of the Galilean group composition $g_1 g_3 g_2$ *on* $(\mathbf{r}, t)$ *from the left is given by*

$$g(\mathbf{r}, t) \;=\; (O\mathbf{r} + t\mathbf{v}_0 + \mathbf{r}_0, t + t_0)\,,$$

for

$$g = g_1(\mathbf{r}_0, t_0) g_3(\mathbf{v}_0) g_2(O) =: g_1 g_3 g_2 \,.$$

> **Exercise.** Write the corresponding transformations for $g_1g_2g_3$, $g_2g_1g_3$ and $g_3g_2g_1$ as well, showing how they depend on the order in which the rotations, boosts and translations are composed. Write the inverse transformation for each of these compositions of left actions.
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> ★

Answer. For translations, $g_1(\mathbf{r}_0, t_0)$, rotations $g_2(O)$ and boosts

$g_3(\mathbf{v}_0)$,

$$\begin{aligned} g_1g_2g_3(\mathbf{r},t) &= \Big(O(\mathbf{r}+t\mathbf{v}_0)+\mathbf{r}_0,\ t+t_0\Big)\,, \\ g_1g_3g_2(\mathbf{r},t) &= \Big(O\mathbf{r}+t\mathbf{v}_0+\mathbf{r}_0, t+t_0\Big)\,, \\ g_2g_1g_3(\mathbf{r},t) &= \Big(O(\mathbf{r}+t\mathbf{v}_0+\mathbf{r}_0),\ t+t_0\Big)\,, \\ g_3g_2g_1(\mathbf{r},t) &= \Big(O(\mathbf{r}+\mathbf{r}_0)+t\mathbf{v}_0,\ t+t_0\Big)\,. \end{aligned}$$

The inverses are $(g_1g_2g_3)^{-1} = g_3^{-1}g_2^{-1}g_1^{-1}$, etc. ▲

Remark 1.2.6 (Decomposition of the Galilean group)
Because the rotations take vectors into vectors, any element of the transformations $g_1g_2g_3$, $g_2g_1g_3$ and $g_3g_2g_1$ in the Galilean group may be written uniquely in the simplest form, as $g_1g_3g_2$.

Thus, any element of the Galilean group may be written uniquely as a rotation, followed by a space translation, a Galilean boost and a time translation. The latter three may be composed in any order, because they commute with each other.

Exercise.
What operations are preserved by the Galilean group?
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Answer. The Galilean group $G(3)$ preserves length and time intervals. (It preserves length, for example, because rotating, translating, or running alongside a ruler does not make it get longer.) The Galilean group also preserves relative angles between vectors, so it preserves the projection (inner product) of vectors. So the Galilean group preserves the results of measuring length, time and relative orientation. ▲

Definition 1.2.7 (Subgroup)
*A **subgroup** is a subset of a group whose elements also satisfy the defining properties of a group.*

Exercise. List the subgroups of the Galilean group that do not involve time. ★

Answer. The subgroups of the Galilean group that are independent of time consist of:

- Space translations $g_1(\mathbf{r}_0)$ acting on $\mathbf{r}$ as $g_1(\mathbf{r}_0)\mathbf{r} = \mathbf{r} + \mathbf{r}_0$.

- Proper rotations $g_2(O)$ with $g_2(O)\mathbf{r} = O\mathbf{r}$ where $O^T = O^{-1}$ and $\det O = +1$. This subgroup is called $SO(3)$, the ***special orthogonal group*** in three dimensions.

- Rotations and reflections $g_2(O)$ with $O^T = O^{-1}$ and $\det O = \pm 1$. This subgroup is called $O(3)$, the ***orthogonal group*** in three dimensions.

- Space translations $g_1(\mathbf{r}_0)$ composed with proper rotations $g_2(O)$ acting on $\mathbf{r}$ as

$$E(O, \mathbf{r}_0)\mathbf{r} = g_1(\mathbf{r}_0)g_2(O)\mathbf{r} = O\mathbf{r} + \mathbf{r}_0\,,$$

  where $O^T = O^{-1}$ and $\det O = +1$. This subgroup is called $SE(3)$, the ***special Euclidean group*** in three dimensions.

- Space translations $g_1(\mathbf{r}_0)$ composed with proper rotations and reflections $g_2(O)$, as $g_1(\mathbf{r}_0)g_2(O)$ acting on $\mathbf{r}$. This subgroup is called $E(3)$, the ***Euclidean group*** in three dimensions.

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Remark 1.2.8 *Spatial translations and rotations do not commute in general. That is, $g_1 g_2 \neq g_2 g_1$, unless the direction of translation and axis of rotation are collinear.*

> **Exercise.** (Matrix representation for $SE(3)$)
> Write the composition law $g_1(\mathbf{r}_0)g_2(O)\mathbf{r} = O\mathbf{r} + \mathbf{r}_0$ for $SE(3)$ as a multiplication of 4×4 matrices. ★

Answer. The special Euclidean group in three dimensions $SE(3)$ acts on a position vector $\mathbf{r}$ as

$$E(O, \mathbf{r}_0)\mathbf{r} = O\mathbf{r} + \mathbf{r}_0\,.$$

This action may be given a 4×4 matrix representation by noticing that the right side arises in multiplying the matrix times the extended vector $(\mathbf{r}, 1)^T$ as

$$\begin{pmatrix} O & \mathbf{r}_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{r} \\ 1 \end{pmatrix} = \begin{pmatrix} O\mathbf{r} + \mathbf{r}_0 \\ 1 \end{pmatrix}.$$

Therefore we may identify a group element of $SE(3)$ with a 4×4 matrix as,

$$E(O, \mathbf{r}_0) = \begin{pmatrix} O & \mathbf{r}_0 \\ 0 & 1 \end{pmatrix}.$$

The group $SE(3)$ has six parameters. These are the angles of rotation about each of the three spatial axes by the orthogonal matrix O with $O^T = O^{-1}$ and the three components of the vector of translations $\mathbf{r}_0$.

The group composition law for $SE(3)$ is expressed as

$$\begin{aligned} E(\tilde{O}, \tilde{\mathbf{r}}_0)E(O, \mathbf{r}_0)\mathbf{r} &= E(\tilde{O}, \tilde{\mathbf{r}}_0)(O\mathbf{r} + \mathbf{r}_0) \\ &= \tilde{O}(O\mathbf{r} + \mathbf{r}_0) + \tilde{\mathbf{r}}_0 \,, \end{aligned}$$

with $(O\,,\,\tilde{O}) \in SO(3)$ and $(\mathbf{r}\,,\,\tilde{\mathbf{r}}_0) \in \mathbb{R}^3$. This formula for group composition may be represented by matrix multiplication from the left as

$$\begin{aligned} E(\tilde{O}, \tilde{\mathbf{r}}_0)E(O, \mathbf{r}_0) &= \begin{pmatrix} \tilde{O} & \tilde{\mathbf{r}}_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} O & \mathbf{r}_0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \tilde{O}O & \tilde{O}\mathbf{r}_0 + \tilde{\mathbf{r}}_0 \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

which may also be expressed by simply writing the top row,

$$(\tilde{O}\,,\,\tilde{\mathbf{r}}_0)(O\,,\,\mathbf{r}_0) = (\tilde{O}O\,,\,\tilde{O}\mathbf{r}_0 + \tilde{\mathbf{r}}_0)\,.$$

The identity element (e) of $SE(3)$ is represented by

$$e = E(I, \mathbf{0}) = \begin{pmatrix} I & \mathbf{0} \\ 0 & 1 \end{pmatrix},$$

or simply $e = (I, \mathbf{0})$. The inverse element is represented by the matrix inverse

$$E(O, \mathbf{r}_0)^{-1} = \begin{pmatrix} O^{-1} & -O^{-1}\mathbf{r}_0 \\ 0 & 1 \end{pmatrix}.$$

In the matrix representation of $SE(3)$, one checks directly that,

$$\begin{aligned} E(O, \mathbf{r}_0)^{-1} E(O, \mathbf{r}_0) &= \begin{pmatrix} O^{-1} & -O^{-1}\mathbf{r}_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} O & \mathbf{r}_0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} I & \mathbf{0} \\ 0 & 1 \end{pmatrix} = (I, \mathbf{0}) = e\,. \end{aligned}$$

In the shorter notation the inverse may be written as

$$(O\,,\,\mathbf{r}_0)^{-1} = (O^{-1}\,,\,-O^{-1}\mathbf{r}_0)\,,$$

and $O^{-1} = O^T$ since the 3×3 matrix O is orthogonal. ▲

Remark 1.2.9 *The inverse operation of $SE(3)$ involves composition of the inverse for rotations with the inverse for translations. This means $SE(3)$ is not a direct product of its two subgroups $\mathbb{R}^3$ and $SO(3)$.*

1.3 Lie group actions of $SE(3)$ & $G(3)$

Group multiplication in $SE(3)$ is denoted as

$$(\tilde{O}\,,\,\tilde{\mathbf{r}}_0)(O\,,\,\mathbf{r}_0) = (\tilde{O}O\,,\,\tilde{O}\mathbf{r}_0 + \tilde{\mathbf{r}}_0)\,.$$

This notation demonstrates the following Lie group actions of $SE(3)$:

1. ***Translations*** in the subgroup $\mathbb{R}^3 \subset SE(3)$ act on each other by vector addition,

$$\mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R}^3 : \ (I\,,\,\tilde{\mathbf{r}}_0)(I\,,\,\mathbf{r}_0) = (I\,,\,\mathbf{r}_0 + \tilde{\mathbf{r}}_0)\,.$$

2. ***Rotations*** in the subgroup $SO(3) \subset SE(3)$ act on each other as

$$SO(3) \times SO(3) \mapsto SO(3) : \ (\tilde{O}\,,\,\mathbf{0})(O\,,\,\mathbf{0}) = (\tilde{O}O\,,\,\mathbf{0})\,.$$

3. Rotations in subgroup $SO(3) \subset SE(3)$ ***act homogeneously*** on the vector space of translations in the subgroup $\mathbb{R}^3 \subset SE(3)$

$$SO(3) \times \mathbb{R}^3 \mapsto \mathbb{R}^3 : \; (\tilde{O}\,,\,\mathbf{0})(I\,,\,\mathbf{r}_0) = (\tilde{O}\,,\,\tilde{O}\mathbf{r}_0)\,.$$

   That is, the action of the subgroup $SO(3) \subset SE(3)$ on the subgroup $\mathbb{R}^3 \subset SE(3)$ maps the $\mathbb{R}^3$ into itself. The translations $\mathbb{R}^3 \subset SE(3)$ are thus said to form a ***normal***, or ***invariant subgroup*** of the group $SE(3)$.

4. Every element of $(O\,,\,\mathbf{r}_0)$ of $SE(3)$ may be represented uniquely by composing a translation acting from the left on a rotation. That is, each element may be decomposed into

$$(O\,,\,\mathbf{r}_0) = (I\,,\,\mathbf{r}_0)(O\,,\,\mathbf{0})\,,$$

   for a *unique* $\mathbf{r}_0 \in \mathbb{R}^3$ and $O \in SO(3)$. Likewise, one may uniquely represent

$$(O\,,\,\mathbf{r}_0) = (O\,,\,\mathbf{0})(I\,,\,O^{-1}\mathbf{r}_0)\,,$$

   by composing a rotation acting from the left on a translation.

Definition 1.3.1 (Semidirect-product Lie group)
A Lie group G that may be decomposed uniquely into a normal subgroup N and a subgroup H such that every group element may be written as

$$g = nh \quad or \quad g = hn \quad (in\ either\ order),$$

for unique choices of $n \in N$ and $h \in H$ is called a ***semidirect product*** *of H and N, denoted in the present convention as*

$$G = H \circledS N\,.$$

Remark 1.3.2 *The special Euclidean group $SE(3)$ is an example of a semidirect-product Lie group, denoted as*

$$SE(3) = SO(3) \circledS \mathbb{R}^3\,.$$

Remark 1.3.3 *The Galilean group in three dimensions $G(3)$ has ten parameters ($O \in SO(3)$, $\mathbf{r}_0 \in \mathbb{R}^3$, $\mathbf{v}_0 \in \mathbb{R}^3$, $t_0 \in \mathbb{R}$). The Galilean group is also a semidirect-product Lie group, which may be written as*

$$G(3) = SE(3) \circledS \mathbb{R}^4 = \Big(SO(3) \circledS \mathbb{R}^3\Big) \circledS \mathbb{R}^4 .$$

That is, the subgroup of Euclidean motions acts homogeneously on the subgroups of Galiliean boosts and time translations $(\mathbf{v}_0, t_0) \in \mathbb{R}^4$ which commute with each other.

Exercise. Compute explicitly the inverse of the Galilean group element $g_1 g_3 g_2$ obtained by representing the action of the Galilean group as matrix multiplication of the extended vector $(\mathbf{r}, t, 1)^T$,

$$g_1 g_3 g_2 \begin{pmatrix} \mathbf{r} \\ t \\ 1 \end{pmatrix} = \begin{pmatrix} O & \mathbf{v}_0 & \mathbf{r}_0 \\ \mathbf{0} & 1 & t_0 \\ \mathbf{0} & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{r} \\ t \\ 1 \end{pmatrix} = \begin{pmatrix} O\mathbf{r} + t\mathbf{v}_0 + \mathbf{r}_0 \\ t + t_0 \\ 1 \end{pmatrix} .$$

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Answer. Write the product $g_1 g_3 g_2$ as

$$g_1 g_3 g_2 = \begin{pmatrix} Id & \mathbf{0} & \mathbf{r}_0 \\ \mathbf{0} & 1 & t_0 \\ \mathbf{0} & 0 & 1 \end{pmatrix} \begin{pmatrix} Id & \mathbf{v}_0 & \mathbf{0} \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{pmatrix} \begin{pmatrix} O & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{pmatrix} .$$

Then, the product $(g_1 g_3 g_2)^{-1} = g_2^{-1} g_3^{-1} g_1^{-1}$ appears in matrix form as

$$\begin{pmatrix} O^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{pmatrix} \begin{pmatrix} Id & -\mathbf{v}_0 & \mathbf{0} \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{pmatrix} \begin{pmatrix} Id & \mathbf{0} & -\mathbf{r}_0 \\ \mathbf{0} & 1 & -t_0 \\ \mathbf{0} & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} O^{-1} & -O^{-1}\mathbf{v}_0 & -O^{-1}(\mathbf{r}_0 - t\mathbf{v}_0) \\ \mathbf{0} & 1 & -t_0 \\ \mathbf{0} & 0 & 1 \end{pmatrix} .$$

▲

Exercise. Check your results from Exercise 1.2 by composing matrix multiplications. ★