CHAPTER 6

THE REYNOLDS AND ENERGY EQUATIONS

6.1. Introduction

In Chapter 5, we discussed the properties of lubricants used in viscous film bearings. Below, we will state, and sometimes derive, some of the equations needed to design these bearings. Where appropriate, we will give some examples of their application. Our approach will be as follows:

- explain the principle of the **hydrodynamic wedge**, both by analogy and by dimensional analysis
- derive Reynolds equation from first principles
- define the energy equation, which accounts for thermal effects in bearings. In subsequent chapters, in order to suit local conditions, modified versions will be employed.

6.2. Reynolds Equation

If you stand by the parallel of a straight part of a steadily flowing river, you will note that the stream velocity is highest in the middle, reducing to zero at the sides. In other words the velocity distribution, of the water, a real fluid, roughly takes a curved shape depicted in Fig. 6.1. An identical velocity distribution would be found whatever position along the bank is chosen, as long as the river geometry is the same.

Additionally, in order for the river to flow at all, its source, on the left of Fig. 6.1, must be at a higher altitude than at its mouth, which is at sea level. The difference in level is called the *head*. It is associated with falling pressures (a negative pressure gradient) driving the stream towards the mouth.

Noting this behavior, let us turn to the next model in Fig. 6.2(a): a simple arrangement of two parallel impervious surfaces, one above the other, long into the paper and separated by a narrow parallel gap. The top surface is stationary and the bottom surface, moving at constant velocity, causes a viscous liquid lubricant to be entrained steadily from left to right.

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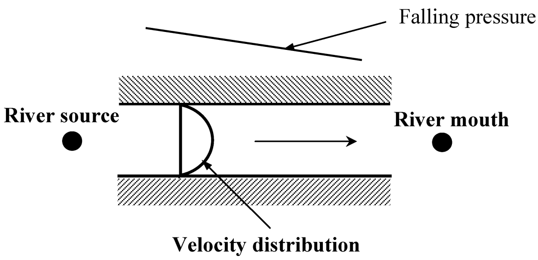

Fig. 6.1. Flowing river.

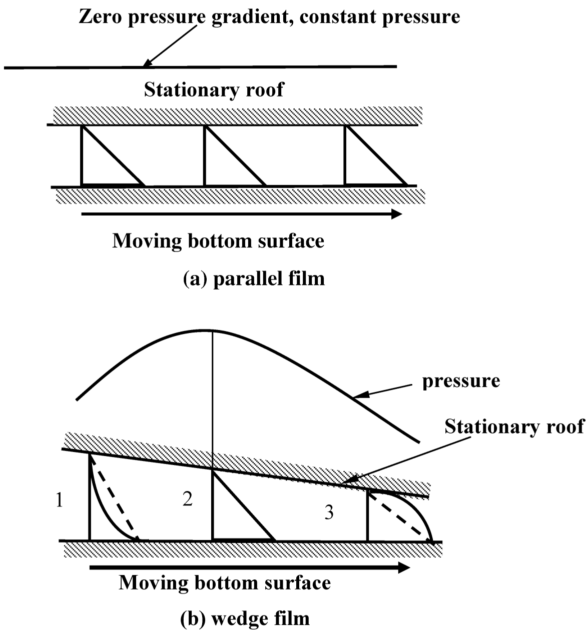

Fig. 6.2. Principle of the wedge film.

The following points should be noted in Fig. 6.2(a):

- Just as in Fig. 6.1, as the flow is constant, the velocity of distribution of the oil is the same whatever position along the gap is chosen.
- As pointed out in Chapter 5, because oil is a Newtonian viscous fluid, it must have the same local velocity as the boundaries wherever it contacts them.

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- Here, the shape of the velocity distribution can only be triangular because, unlike in Fig. 6.1, there is no pressure source, making the pressure gradient zero throughout.

Let us now change the gap shape into a narrowing wedge of small angle, with the same boundary velocity as for the parallel case (Fig. 6.2(b)). Clearly, maintaining the original triangular velocity distributions of Fig. 6.2(a), would infringe continuity. Therefore, the local velocities at the three positions shown must alter so as to maintain the same area under each distribution. This can only be achieved by modifying the oil velocity distributions. In position 1; the widest part of the gap, the distribution bulges backwards while it bulges forwards in position 3, the narrowest part. Note that the *forward* bulge of the velocities in position 3 must be associated with *falling* pressures, as is the distribution in Fig. 6.1. The reverse is true in position 1. Therefore, there must be one point where the velocity distribution is triangular. This is the position 2 and, from Fig. 6.2, there must be a pressure maximum at that point.

The presence of a rising and falling pressure distribution, along the hydrodynamic wedge width, suggests that it must possess **load capacity**. This is a characteristic of all hydrodynamic bearings, because a load capacity is their *raison d'etre*. Like the *head* in a flowing river, an external power source is still needed for driving the moving surface against the retarding friction force of the oil. Incidentally, the same behavior results if the narrowing gap is curved, as long as the inclination angles remain small.

Alternatively, a bearing with *parallel and stationary* surfaces could have been designed if an oil pump was available. Try sketching a possible axisymmetric design that can accommodate a vertical load.

In the next section we will put some algebra into the **hydrodynamic wedge** concept, solely from knowledge we have acquired so far. Also, from now onwards we will often use the general words, **hydrodynamic bearing** or **wedge pair** to explain the wedge concept.

6.3. Reynolds Equation by Dimensional Analysis

Having found the forms of the velocity disribution profiles and the pressure distribution over a hydrodynamic bearing. We will go one step further by finding a mathematical relationship between the variables, solely by dimensional analysis. This is a useful technique, because it helps you to delve into the physics of the problem.

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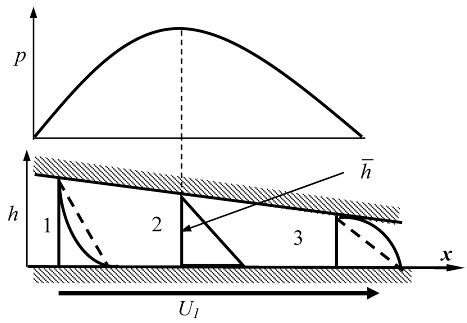

Fig. 6.3. Reynolds equation by dimensional analysis.

Referring to Fig. 6.3, let the volume flow per unit length,* q_x, be of constant density and constant viscosity, η. Recall that, from Sec. 6.2, we discovered that the velocity distribution in a hydrodynamic bearing has two components causing, the total flow, q_x, also to have two parts:

- one due to the velocity of the bottom surface, U_1 (called the **Couette flow**)
- the other from the rising or falling pressure distribution. (called the **Poiseuille flow**, $f(p)$).

The aim of the dimensional analysis is to obtain $f(p)$ and q_x in terms of the other variables: η, U_1 and $h(x)$. This procedure will go part of the way towards a full solution. The area under a velocity profile in Fig. 6.3 is q_x. It can be written as:

$$q_x = \frac{U_1 h}{2} - f(p), \tag{6.1}$$

where $q_x = \frac{\text{total flow rate}}{\text{wedge length into the paper}}$, with dimensions of $\frac{L^3}{TL} = \frac{L^2}{T}$.

For dimensional homegenity in Eq. (6.1), $f(p)$ also has dimensions L^2/T.

As we have reasoned in Sec. 6.2, $f(p)$ is a function of the **pressure gradient** $(\frac{dp}{dx})$, (not magnitude). In addition, it must also depend on the

*Throughout the book, bearing *length* means here the bearing dimension into the paper. This is in keeping with traditional journal bearing terminology, its length being the axial length into the paper.

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local h and η. Therefore, let:

$$f(p) = k\left(\frac{dp}{dx}\right)^a h^b \eta^c, \tag{6.2}$$

where a, b and c are unknown indices and k is an unknown constant. Employing a dimensional analysis with units of *force, length and time*, we can write:

$$\left(\frac{dp}{dx}\right)^a (h)^b (\eta)^c \equiv \left(\frac{F}{L^2 L}\right)^a L^b \left(\frac{F}{L^2}T\right)^c = \frac{L^2}{T}.$$

Equate indices of force: $F^a F^c = 0$ $\quad \therefore a = -c$
and of time: $T^c = T^{-1}$ $\quad \therefore c = -1,$ giving $a = 1$
and of length: $\frac{1}{L^3} L^b L^2 = L^2$ $\quad \therefore b = 3$
Hence:

$$f(p) = k\frac{h^3}{\eta}\frac{dp}{dx}, \quad \text{so} \quad q_x = \frac{U_1 h}{2} - k\frac{h^3}{\eta}\frac{dp}{dx}$$

We can also state that at some point along the bearing, as yet unknown, $\frac{dp}{dx} = 0$. Let this be at $h = h_c$ (station 2, Fig. 6.3).

There, $q_x = \frac{U_1 h_c}{2}$ and as q_x is constant:

$$\frac{U_1 h_c}{2} = \frac{U_1 h}{2} - k\frac{h^3}{\eta}\frac{dp}{dx}, \tag{6.3}$$

or, rearranging Eq. (6.3):

$$\frac{dp}{dx} = \frac{U_1 \eta}{2k}\left(\frac{h - h_c}{h^3}\right). \tag{6.4}$$

A full solution gives k as 1/12. One integration of Eq. (6.4), yields:

$$p = 6U_1\eta \int \frac{h - h_c}{h^3} dx + const. \tag{6.5}$$

h_c and *const.* are integration constants that need 2 boundary conditions along the film to find their values. Equation (6.4) is called the one dimensional Reynolds equation. A full three dimensional solution will now be given below.

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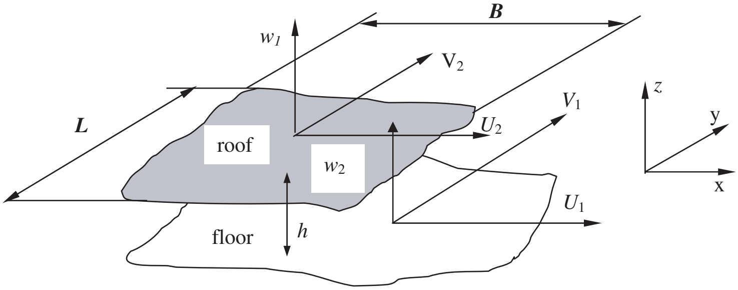

Fig. 6.4. Two generalized surfaces, in relative motion, bounding an oil film.

6.4. Derivation of Reynolds Equation in Three Dimensions

As we did in Sec. 6.3, some realistic assumptions are needed to derive Reynolds equation. Even though we have not yet designed a hydrodynamic bearing, lets assume firstly that the film is thin, so that in profile it has a large aspect ratio (width†/mean film thickness) is typically 1000:1). As an example, for a thrust bearing, the minimum film thickness can be 25 μm with a wedge angle 0.075°. This implies that the film thickness dimensions, $h(x)$, are much less than those defining its top and bottom surfaces, B and L, as illustrated in Fig. 6.4. We may also neglect the film weight (gravity forces).

The differential equation governing pressure distribution in a Newtonian lubricant film was first obtained by Reynolds.[1] It can be derived from the full Navier Stokes (NS) equations[2] by making simplifying assumptions at appropriate points in the analysis. The finished article is a perfectly generalized version of Reynolds equation. However, in our case we will use the more direct engineeing approach: by making simplifying assumptions at the start of the analysis. This approach offers more insight into the physics of the equation.

Assumptions made in the direct derivation of Reynolds equation:

(1) the oil film has negligible mass (gravity forces neglected),
(2) because its so thin, assume pressure is constant across the film (z direction),
(3) no slip at the boundaries (Newtonian fluid),
(4) lubricant flow is laminar (low Reynolds numbers),

†Remember, *width* is measured n the x direction.

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(5) Inertia and surface tension forces are negligible compared with viscous forces,
(6) Because it is thin, shear stresses and velocity gradients are only significant across the film (z direction),
(7) The lubricant is Newtonian (high shear rates are not present),
(8) The lubricant viscosity is constant across the film (z direction),
(9) The boundary surfaces (roof and floor in Fig. 6.4) follow some designated geometry but are always at low angles to each other.

6.4.1. *Equilibrium of forces on a lubricant element*

Referring to Fig. 6.5, let the two bounding surfaces have perfectly general motion defined by their velocity vectors. Consider the local forces acting on a lubricant element, defined at x, y, z in a column of thickness, h. The element center has velocity components u, v, w.

Neglecting shear stress and velocity gradients in the x and y directions (Assumption 6) we have:

$$\sum F_x = 0,$$

$$\therefore \frac{\partial \tau_{xz}}{\partial z} = \frac{\partial p}{\partial x}. \tag{6.6}$$

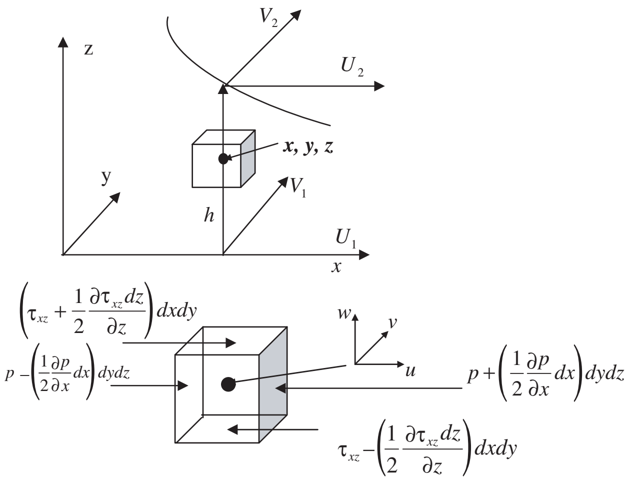

Fig. 6.5. Forces on an element.

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Here, τ_{xz} means shear stress in the x direction in a plane having z as its normal.

Similarly:

$$\sum F_y = 0$$

$$\therefore \frac{\partial \tau_{yz}}{\partial z} = \frac{\partial p}{\partial y}. \tag{6.7}$$

Also, from Eq. (5.1):

$$\tau_{xz} = \eta \frac{\partial u}{\partial z}, \tag{6.8}$$

$$\tau_{yz} = \eta \frac{\partial v}{\partial z}. \tag{6.9}$$

Combining Eq. (6.6) with (6.8), and Eq. (6.7) with (6.9):

$$\frac{\partial}{\partial z}\left[\eta \frac{\partial u}{\partial z}\right] = \frac{\partial p}{\partial x}, \tag{6.10}$$

$$\frac{\partial}{\partial z}\left[\eta \frac{\partial v}{\partial z}\right] = \frac{\partial p}{\partial y}. \tag{6.11}$$

6.4.2. *Velocity distribution*

Let η be invariable in the z direction (Assumption 8). Also, let $\partial p/\partial x$ and $\partial p/\partial y$ not vary with z (Assumption 2). For the x direction, integrate Eq. (6.10) twice with respect to z:

$$\eta u = \frac{\partial p}{\partial x}\frac{z^2}{2} + cz + d.$$

We need two boundary conditions to find the constants c and d. These are:

$$\text{At } z = h, \quad u = U_2 \quad \text{and at} \quad z = 0, \quad u = U_1.$$

Solving for the constants we get:

$$u = \frac{1}{2\eta}\frac{\partial p}{\partial x}\left(z^2 - zh\right) + \frac{z}{h}\left(U_2 - U_1\right) + U_1. \tag{6.12}$$

Similarly, for the y direction, integrating Eq. (6.11) twice:

$$v = \frac{1}{2\eta}\frac{\partial p}{\partial y}\left(z^2 - zh\right) + \frac{z}{h}\left(V_2 - V_1\right) + V_1. \tag{6.13}$$

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Just as we surmised from the wedge-shape study above and found also by dimensional analysis, both Eqs. (6.12) and (6.13) describe velocity distributions composed of two parts. There is a parabolic part due to the pressure gradient (Poisseuille flow) and a linear part due to the boundary surface velocities (Couette flow).

6.4.3. *Mass continuity*

To complete our full derivation of Reynolds equation, we must invoke **mass continuity**. This states that there is the same mass of fluid per second entering a column of oil height, h, as leaving it. Here, we do not have to assume that the fluid is of constant density in the x and y directions. Referring to Fig. 6.6, let m_x and m_y be the mass flows per unit width in the x and y directions at the column center.

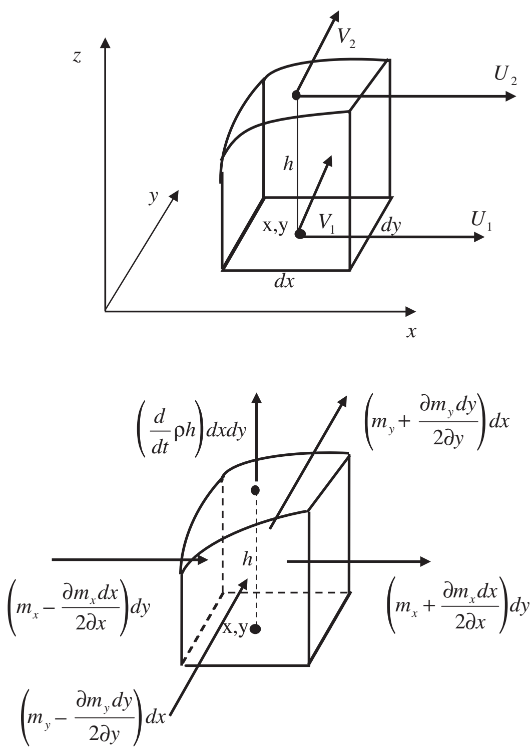

Fig. 6.6. Flow through a column.

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The net mass flow out of the column in the z direction is $(d/dt)\rho h dxdy$ and the mass flows in the x and y directions are, by definition:

$$m_x = \rho q_x = \rho \int_0^h u dz, \qquad (6.14)$$

$$m_x = \rho q_x = \rho \int_0^h u dz. \qquad (6.15)$$

Also, from Fig. 6.6, equating the flows entering the column to those leaving it:

$$\frac{\partial m_x}{\partial x} + \frac{\partial m_y}{\partial y} + \frac{\partial}{\partial t}(\rho h) = 0. \qquad (6.16)$$

Additionally, from Eqs. (6.12)–(6.14) and (6.15):

$$m_x = -\rho \int_0^h \left[\frac{1}{2}\eta \frac{\partial p}{\partial x}(z^2 - zh) + \frac{z}{h}(U_2 - U_1) + U_1\right] dz. \qquad (6.17)$$

Therefore:

$$m_x = -\frac{\rho h^3}{12\eta}\left[\frac{\partial p}{\partial x}\right] + (U_1 + U_2)\left[\frac{\rho h}{2}\right]. \qquad (6.18)$$

Similarly, in the y direction:

$$m_y = -\frac{\rho h^3}{12\eta}\left[\frac{\partial p}{\partial y}\right] + (V_1 + V_2)\left[\frac{\rho h}{2}\right] \qquad (6.19)$$

Equations (6.18) and (6.19) give the mass throughputs of the lubricant per unit length in the x and y directions respectively. Like Eqs. (6.12) and (6.13) each of these is composed of a pressure induced term (Poiseuille flow) and a boundary velocity induced term (Couette flow). If there is only one surface velocity, say in the x direction, then $V_1 = V_2 = 0$. In Eq. (6.19), however, a mass flow, m_y, still occurs in that direction. This now is due only to the pressure gradient, $\frac{\partial p}{\partial y}$, causing some lubricant to flow transversely

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from a high pressure to a low pressure region in that direction. This action in a bearing is called *side leakage*.

Substitute Eqs. (6.18) and (6.19) into Eq. (6.16) and re-arrange the order so that the pressure induced terms only are on the left hand side. The result is the full **Reynolds equation**:

$$\frac{\partial}{\partial x}\left[\frac{\rho h^3}{\eta}\frac{\partial p}{\partial x}\right]+\frac{\partial}{\partial y}\left[\frac{\rho h^3}{\eta}\frac{\partial p}{\partial y}\right] = 6\left\{\frac{\partial}{\partial x}[\rho h(U_1+U_2)]+\frac{\partial}{\partial y}[\rho h(V_1+V_2)]+2\frac{d}{dt}(\rho h)\right\}. \quad (6.20)$$

Equation (6.20) is the fundamental equation of fluid film lubrication theory, with units of $\mathrm{kg\,m^{-2}s^{-1}}$. If the density, ρ, is constant, it cancels out leaving terms of dimension $\mathrm{m\,s^{-1}}$.

It is the full Reynolds equation in three dimensions for compressible or incompressible flow of a Newtonian fluid. On the left hand side are the Poiseuille pressure induced terms. The right hand side is composed of Couette terms which, from left to right, are divided into **wedge** and **squeeze** components. Equation (6.20) accounts for flow components in the x and y directions. Neither boundary surface need have a uniform velocity vector parallel to the xy plane because the components are respectively situated within the differentials on the right hand side, $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. The same applies to the right hand side, where ρ and η are again within the differential operators. Both may vary with x and y because of their sensitivities to pressure (we have seen that it rises and falls) and/or temperature. Finally, the squeeze term, $\frac{\partial}{\partial t}(\rho h)$, need not be uniform over the region of pressure. If the squeeze term is expanded, it can be written as: $\rho(w_1 - w_2) + h\frac{d\rho}{dt}$ where $(w_1 - w_2) = \frac{dh}{dt}$ and w_1 and w_2, are respectively the roof and floor velocities, from whatever cause.

6.5. Simplifications of Reynolds Equation

Generally, we will be dealing with simplified versions of Eq. (6.20). For the two examples below, assume:

- the oil density and viscosity are constant.
- the roof and floor of the film are non-porous and have no normal velocity components.
- the surface velocities are in the x direction only.

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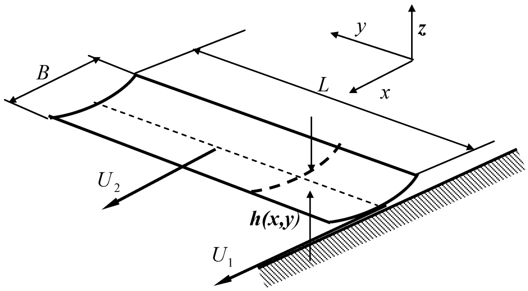

Fig. 6.7. Long bearing $L \gg B$.

6.5.1. *Long bearing*

As a simplification we assume that the transverse length of the bearing, L, (y direction) is effectively infinite. In practice we call this a **long bearing** (L much greater than the width, B, in the x direction). The long transverse length assumption makes the pressure distribution uniform in the y direction, except close to the edges, where the pressures must drop to zero there. The shape is illustrated in Fig. 6.7.

In this case, dp/dy can be neglected compared with dp/dx, so Eq. (6.20) becomes:

$$\frac{d}{dx}\left(\frac{h^3}{\eta}\frac{dp}{dx}\right) = 6(U_1 + U_2)\frac{dh}{dx}. \tag{6.21}$$

Equation (6.21) can be integrated with respect to x:

$$h^3\frac{dp}{dx} = 6(U_1 + U_2)\eta h + C,$$

where C is an integration constant.
Let $h = h_c$ (as yet unknown) at $dp/dx = 0$.
Equation (6.21) modifies to:

$$\frac{dp}{dx} = 6(U_1 + U_2)\eta\left[\frac{h - h_c}{h^3}\right]. \tag{6.22}$$

Equation (6.22) is the differential form of Eq. (6.5) we obtained by dimensional analysis. It also confirms the multiplying constant as $k = 1/12$. This equation will be used frequently throughout the book. The **long bearing assumption** is reasonably accurate if $L/B > 3$. Below is an example.

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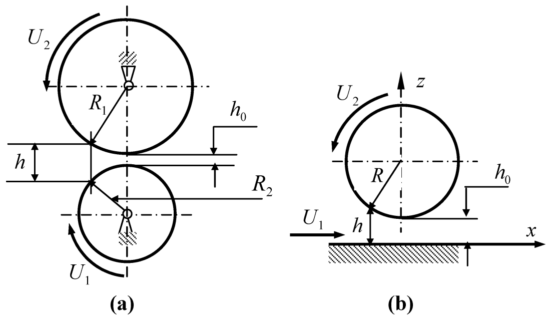

Fig. 6.8. Line contact geometry (a) two discs (b) equivalent contact pressure distribution.

6.5.2. *Long bearing approximation for rigid cylinders*

An important application of Eq. (6.22) is the lubrication of long rigid cylinders where we will need to use additional boundary conditions in Reynolds equation for a complete solution. The model below is a useful precursor to the analysis in Chapter 10, where the bounding surfaces are assumed to be elastic.

Figure 6.8(a) shows two rigid cylinders, loaded on their circumferences, in rolling-sliding lubricated contact, producing a **convergent-divergent wedge film**. Just as the one in Fig. 3, for static contacts, this arrangement is called a **hydrodynamic line contact**, even if the surfaces are distorted. Let us first find the pressure distribution created by the wedging action of the surfaces.

6.5.3. *Line contact pressure distribution*

We will make the following assumptions:

(1) The discs are considered long in the y direction, so that there is no transverse flow except near their ends,
(2) $R \gg h$,
(3) Conditions are isoviscous throughout, so that $\eta = \eta_0. =$ constant.

Referring to Fig. 6.8 (b) and Eq. (3.8), in order to simplify the coordinate system let the discs be replaced by an equivalent disc of reduced radius, R, in contact with a plane, where $1/R = 1/R_1 + 1/R_2$. In addition, in keeping with normal practice for lubricated concentrated contacts, let the mean velocity of the Couette flow component be $U = \frac{U_1+U_2}{2}$ (called the

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entrainment velocity). Equation (6.22) therefore becomes:

$$\frac{dp}{dx} = 12U\eta_0 \frac{h - h_c}{h^3}. \tag{6.23}$$

Also, from Assumption 3, the film shape can be written as:

$$h \approx h_0 + \frac{x^2}{2R}. \tag{6.24}$$

Equation (6.24) will be used in Chapter 10 for counterformal EHL line contacts.

It is usual practice to employ dimensionless groups, so let:

$$\bar{x} = \frac{x}{\sqrt{2Rh_0}}, \quad \bar{h} = \frac{h}{h_0},$$

Making this substitution, Eq. (6.24) becomes:

$$\bar{h} = 1 + \bar{x}^2,$$

and letting $\bar{p} = \frac{h_0^{3/2} p}{12U\eta_0\sqrt{2R}}$, with $\bar{x}_c$ the $\bar{x}$ coordinate when $\bar{h} = \bar{h}_c$, we get:

$$\frac{d\bar{p}}{d\bar{x}} = \frac{\bar{x}^2 - \bar{x}_c^2}{(1 + \bar{x}^2)^3}. \tag{6.25}$$

Integrating Eq. (6.25) with respect to $\bar{x}$:

$$\bar{p} = \int \frac{\bar{x}^2 d\bar{x}}{(1 + \bar{x}^2)^3} - \bar{x}_c^2 \int \frac{d\bar{x}}{(1 + \bar{x}^2)^3}. \tag{6.26}$$

These are standard integrals can be found using Math CAD or from Ref. 3. The solution comes to:

$$\bar{p} = \left[\frac{-\bar{x}}{4(1 + \bar{x}^2)^2} + \frac{\bar{x}}{8(1 + \bar{x}^2)} + \frac{1}{8} tg^{-1}(\bar{x})\right] - \bar{x}_c^2 \left[\frac{x}{4(1 + \bar{x}^2)^2} + \frac{3}{8}\frac{\bar{x}}{(1 + \bar{x}^2)} + \frac{3}{8} tg^{-1}\bar{x}\right] + C_1. \tag{6.27}$$

We will need two additional boundary conditions to determine $\bar{x}_c$ and C_1. (Remember in deriving Eq. (6.23) we started by using boundary conditions at the top and bottom of the film (z direction). By this stage the boundary

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values are in the x direction. One pair, called the **Full Sommerfeld**[4] condition, discussed in Ref. 3, is:

$$p = 0 \quad \text{at} \quad \bar{x} = -\infty.$$

The condition at $\bar{x} = -\infty$ is called a **fully flooded** or **drowned inlet**. Inserting the above two boundary conditions into Eq. (6.27) we get after some manipulation and reference to standard integrals:

$$\bar{p} = \frac{-\bar{x}}{3(1 + \bar{x}^2)^2}. \tag{6.28}$$

Equation (6.28) has an antisymmetric shape producing zero load capacity because of the positive and pressure loops of equal area. One alternative model is to ignore the negative pressures. It is called the **Half Sommerfeld boundary condition**[4] and discussed in Ref. 3. This is curve (b) in Fig. 6.9, obtained from Eq. (6.28) with $p = 0$ when $\bar{x} \leq 0$. Although it gives reasonable approximate answers for the pressure distribution, the abrupt change of pressure gradient to zero at $x = 0$, cannot occur because flow continuity is contravened (The Poiseuille flow component has suddenly

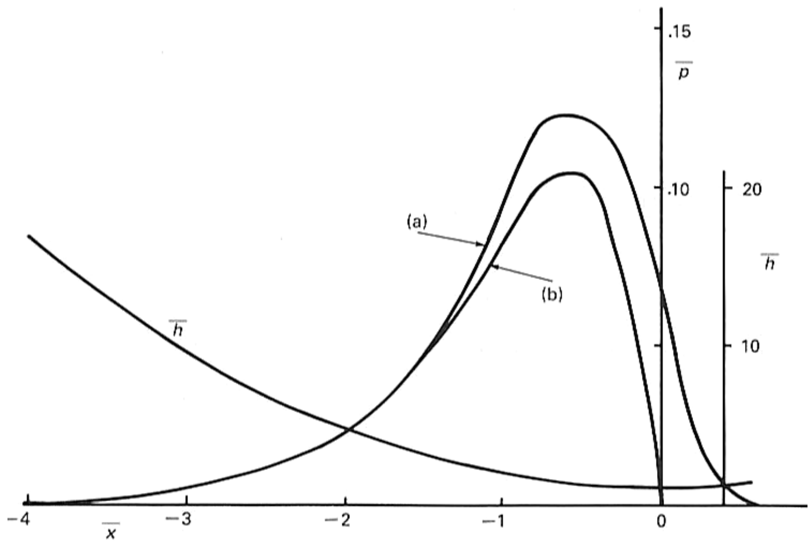

Fig. 6.9. Line contact pressure distribution for a flooded inlet distance. (a) Reynolds exit condition: (b) Half Sommerfeld exit condition.

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vanished, although the Couette flow component continues across the z axis). Swift and Steiber[5] have suggested a more realistic boundary condition that does not contravene flow continuity, known as the **Reynolds exit boundary condition** (curve (a) in Fig. 6.9). It is often the one of choice in numerical solutions, especially when there are high loads, such as under EHL conditions (see Chapter 10). The Reynolds exit boundary condition states that the pressure exit boundary is where $p = dp/dx = 0$ (a little beyond $x = 0$). Experiments show that with Reynolds condition where film rupture occurs, the remaining Couette flow component has to expand into a widening gap, causing the flow to break up into oil carrying partitions (fingers) separated by air gaps (cavities). There are some illustrations of these in Ref. 6.

6.5.4. *Line contact load*

Using the half Sommerfeld boundary condition, integration of Eq. (6.28) between $x = -\infty$ and 0, produces a load per unit length of[6]:

$$\frac{W}{L} = \frac{4U\eta_0 R}{h_0}. \tag{6.29}$$

Using the Reynolds exit boundary condition, the load per unit length is[6]:

$$\frac{W}{L} = \frac{4.9U\eta_0 R}{h_0}. \tag{6.30}$$

Thus, for **rigid line contacts** with flooded inlets, the half Sommerfeld boundary condition produces a reasonable approximation to the Reynolds boundary condition load. However, is this predicted film thickness realistic? To give us some idea of scale, let $R = 0.0127\,\text{m}$, $W = 5000\,\text{N}$, $L = 0.025\,\text{m}$, $U = 1\,\text{m/s}$, $\eta_0 = 0.1\,\text{Nm}^{-2}\text{s}$. The least film thickness, using either expression is roughly $h_0 = 3 \times 10^{-8}\,\text{m}$. This low value is unrealistic in engineering terms, because it is below normal surface roughnesses. What happens in practice is that the counterformally contacting surfaces distort elastically and also the viscosity increases with pressure, producing realistic film thicknesses that are ten times this value This condition is tackled in Chapter 10, when dealing with EHL and in Chapter 8, for journal bearings, the boundary conditions discussed above are again applied to rigid surfaces that are in highly conformal contact.

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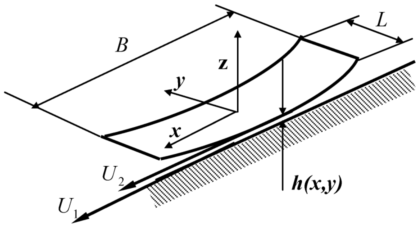

Fig. 6.10. Narrow bearing.

6.5.5. *Narrow bearings*

At the other extreme, if the bearing is assumed to be narrow, as in Fig. 6.10, $B << L$, we can assume now that $\partial p/\partial x$ can be neglected compared with $\partial p/\partial y$ With the above assumption, Eq. (6.20) becomes:

$$\frac{d}{dx}\left\{h^3\frac{dp}{dy}\right\} = 6(U_1+U_2)\eta\frac{dh}{dx}. \tag{6.31}$$

Another assumption we must make for this simple bearing, is that $h \neq f(y)$. We can, therefore, remove h from the bracket to give:

$$\frac{d^2p}{dy^2} = 6(U_1+U_2)\eta\frac{dh}{dx}\frac{1}{h^3}. \tag{6.32}$$

Integrating Eq. (6.31) with respect to y and noting that dh/dx does not vary with y:

$$\frac{dp}{dy} = 6\left\{(U_1+U_2)\eta\frac{dh/dx}{h^3}\right\}y + C_1. \tag{6.33}$$

Integrating again with respect to y:

$$p = 6\left\{(U_1+U_2)\eta\frac{dh/dx}{h^3}\right\}\frac{y^2}{2} + C_1y + C_2. \tag{6.34}$$

For the long bearing, we have generated two integration constants that can be obtained in the following way: Because of symmetry about the x axis, at the bearing ends: $p = 0$ along $y = \pm L/2$ and also at the center $dp/dy = 0$ along $y = 0$, yielding: $C_1 = 0, C_2 = -3(U_1+U_2)\eta\frac{dh/dxL^2}{h^3}\frac{L^2}{4}$.
Thus:

$$p = 3(U_1+U_2)\eta\frac{dh/dx}{h^3}(y^2 - (L^2/4)). \tag{6.35}$$

Note that in Eq. (6.35), dh/dx has not been stipulated, because only certain film shapes are eligible for short bearing solutions. The reason is that, along

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both the x and y direction boundaries of the film, the pressure must be zero. This is fine along the x direction boundaries with the rising and falling symmetrical parabola shown, but along the y direction boundaries of the film pressure can only be zero if $dh/dx = 0$ there. Thus, Eq. (6.35) must always have a zero slope in the x direction along the y direction boundaries. Fortunately, journal bearings have this characteristic so that Eq. (6.27) offers a simple design approximation, as we shall see later. On the other hand, for a straight wedge shaped bearing (flat roof and floor) Eq. (6.27) cannot be used. Finally, the narrow bearing assumption for certain bearing geometries can be used in practice with reasonable accuracy only if $L/B < 0.5$.

6.5.6. *Squeeze film bearings*

Equation (6.20) also applies to another type of bearing behavior depending on the **squeeze film effect**. If we place a flat plate onto a uniform thin film of oil, it will sink down slowly. The more viscous the oil, the more slowly it will sink. This viscous resistance is governed by the squeeze film effect. It is accommodated by the last term on the right hand side of Eq. (6.20). Noting that at any instant h is constant for a flat plate, and assuming ρ is invariable, Eq. (6.21) becomes:

$$\frac{\partial}{\partial x}\left[\frac{h^3}{\eta}\frac{\partial p}{\partial x}\right] + \frac{\partial}{\partial y}\left[\frac{h^3}{\eta}\frac{\partial p}{\partial y}\right] = 12\frac{dh}{dt}. \tag{6.36}$$

For a circular flat plate or a journal bearing, Reynolds equation is better expressed in polar coordinates. We will tackle a similar problem in Chapter 12 when we deal with alternating loads on journal bearings.

6.6. Rolling Contacts

Another important modification of Eq. (6.20) applies to any surface of revolution rolling without slip on a *stationary* plane covered by a lubricant film, such as the cylinder in Fig. 6.11(a). This is the same as the example above for the line contact of rigid discs in Sec. 6.5.3. However, the values to be assigned to U_1, are not immediately obvious. A full discussion of how this is done is given by Gohar in Ref. 6, but a quick solution can be found. In Fig. 6.11(a), if we put a backward velocity, $-U_1$, onto the cylinder center,

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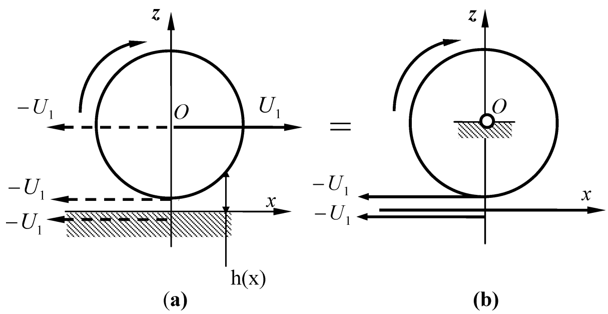

Fig. 6.11. Rolling cylinder.

the film roof and the floor (all dotted), we freeze point O, thus making the film roof and floor velocities at A both $-U_1$, as in (b).

Equation (6.22) therefore becomes for a rolling contact:

$$\frac{dp}{dx} = 12U_1\eta\left[\frac{h - h_c}{h^3}\right]. \tag{6.37}$$

Note again that Fig. 6.11 represents a **convergent-divergent wedge**, with the inlet part on the right hand side in both (a) and (b). We will frequently encounter this form of fluid film kinematics, when we deal with lubricated rolling element bearings.

6.7. The Energy Equation

The other fundamental equation needed is the **Energy equation**, where a full proof is given in Ref. 3. We will present it here in two dimensional form without proof, at a level suitable for our purposes. It is, for a Newtonian fluid:

$$\underbrace{\nu u\theta\frac{\partial p}{\partial x}}_{\textbf{compressive heating}} + \underbrace{\eta\left(\frac{\partial u}{\partial z}\right)^2}_{\textbf{viscous heating}} = \underbrace{\rho u c_p\frac{\partial \theta}{\partial x}}_{\textbf{convection cooling}} - \underbrace{k_t\frac{\partial^2\theta}{\partial z^2}}_{\textbf{conduction cooling}}, \tag{6.38}$$

where $\theta(x)$ is the temperature rise of the oil from inlet, ν is its coefficient of thermal expansion, c_p its specific heat (at constant pressure), and k_t its thermal conductivity.

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The following points should be noted in connection with Eq. (6.38):

- The **compression heating** term on the left hand side, caused by the pressure distribution, is relatively insignificant and therefore will be ignored in our subsequent simple analysis. The viscous heating term comes from shearing in the x direction across the film.
- On the right hand side of the equation, the convection cooling term, carries some of the heat away through the flow in the x direction, while the conduction cooling term carries some away in the z direction across the solid boundaries. This term was discussed in Sec. 4.4.
- When later we deal with heat transfer in hydrodynamic bearings, the **viscous heating** and **convection cooling** terms are the most important because of the relatively thick films encountered.
- On the other hand, when we deal with **Elastohydrodynamic** contacts **(EHL)**, the viscous heating and conduction cooling terms are the most important because the oil films are much thinner.
- For our purposes, Eq. (6.38) is simplified considerably if we assume that the bearing film is sensibly parallel and long in the y direction, with the dominant heating from Couette flow (caused only from the x direction velocities of the boundaries).

With the above assumptions Eq. (6.38) becomes:

$$\eta \left(\frac{\partial u}{\partial z} \right)^2 = \rho u c_p \frac{\partial \theta}{\partial x} + k_t \frac{\partial^2 \theta}{\partial z^2}. \tag{6.39}$$

6.7.1. *Significance of terms in the energy equation*

Firstly, we must determine the relative significance of the right hand side terms of Eq. (6.39), in relation to the film thickness. With the above assumptions, we have Fig. 6.12.

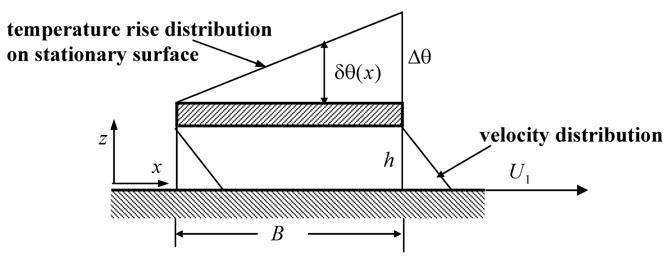

Fig. 6.12. Thermal effects.

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6.7.2. *Convected heat only*

Dealing with the **convection cooling** term only in Eq. (6.26) we will use that term to derive an order of magnitude expression for the heat convected away. Assume that in this case the bounding surfaces are completely insulated so that all the heat is carried away by convection along the film. Let the average fluid velocity be $U_1/2$ and the maximum average temperature rise in the film be $\Delta\theta/2$. Then, at position x, the convected heat flow out across distance dx is:

$$\rho c_p \frac{d\theta}{dx} dx \int_0^h u dz = \rho c_p \left(\frac{d\theta}{dx}\right)\left(\frac{U_1 h}{2}\right) dx.$$

But at $x = B$, $d\theta/dx = \Delta\theta/2B$. Therefore, total convected heat flow out of the bearing end is

$$\frac{\rho c_p U_1 h}{2}\left(\frac{\Delta\theta}{2B}\right)\int_0^B dx = \frac{\rho c_p U_1 h \Delta\theta}{4} = \left(\frac{U_1 \rho h}{2}\right) \times c_p \times \frac{\Delta\theta}{2}. \qquad (6.40)$$

The last term in Eq. (6.40) is an easy way of remembering that the convective heat flow rate is: mass flow per second × specific heat × temperature rise.

6.7.3. *Conducted heat only*

If all heat is removed by **conduction**, an order of magnitude solution, assumes that the temperature gradient *across* the film varies linearly for each value of x, rising from zero on the moving surface (along $z = 0$) to $\delta\theta/h$ on the stationary top surface, giving a parabolic temperature distribution across the film. Therefore, integrating with respect to z, the heat flow rate into the top surface through a column of width dx and height, h, is:

$$k_t dx \int_0^h (d^2\theta/dz^2) dz = k_t dx(\delta\theta/h). \qquad (6.41)$$

Substituting $\delta\theta = x\Delta\theta/B$ into Eq. (6.41), the total conducted heat flow through the whole top surface is:

$$\frac{k_t \Delta\theta}{Bh}\int_0^B x dx = \frac{k_t \Delta\theta B}{2h}. \qquad (6.42)$$

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Table 6.1. Diffusivity and Peclet numbers.

Fluid	Oil	Water	Air (STP)
Diffusivity (κ) (m^2/s)	8.96×10^{-8}	1.4×10^{-7}	2.180×10^{-5}
Peclet number (Pe)	14.92	9.346	0.061

6.7.4. *Heat flow ratio*

Therefore, from Eqs. (6.40) and (6.42), the ratio of convected to conducted heat (defined here as the fluid **Peclet number**) is given by the relationship:

$$Pe = \frac{\left(\frac{U_1 h}{4} c_p \rho \Delta\theta\right)}{\left(\frac{\Delta\theta k_t B}{2h}\right)} = \frac{\frac{U_1 h^2}{2B}}{\frac{k_t}{\rho c_p}}. \tag{6.43}$$

The bracketed group of material constants on the right hand side of Eq. (6.43) is called the **thermal diffusivity** (κ) of the fluid. For various fluids, Table 6.1 gives values of thermal diffusivity and Peclet Numbers for a typical high speed long journal of diameter $D = 0.0076\,\text{m}$ ($B = \pi D$), surface speed $U_1 = 61\,\text{ms}^{-1}$ and radial clearance $h = 10^{-4}\,\text{m}$ bearing (see Chapter 8).

Note that k_t has units of: $\text{kJ}\,\text{m}^{-1}\text{s}^{-1}(^\circ\text{K})^{-1}$ and σ has units of: $\text{kJ}\,\text{Kg}^{-1}(^\circ\text{K})^{-1}$.

For the three fluids shown, oil is the best medium for carrying some of the heat away by convection, while air is best for transfer by conduction.

Had the model been two involute gear teeth meshing in oil, typically of contact width 0.002 m, total surface speed 10 m/s and an average film thickness of 1 micron, then Pe = 0.0279, which is about 1/500 of that of the journal bearing above.

6.8. Closure

In this chapter we have discussed the significance of the hydrodynamic wedge in Lubrication theory. We then derived Reynolds equation, applying it to both long and narrow geometries, together with the various boundary conditions employed. Finally, the relative significance of the terms composing the Energy equation was analyzed in relation to heat transfer. Some of these initial results will be applied to the design of bearings in subsequent chapters. The next chapter will analyse the thrust bearing and its practical applications.

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References

1. Reynolds, O., *Phil. Trans.* **177** (1886) 157–244.
2. Middleman, S., *An Introduction to Fluid Dynamics*, John Wiley & Sons NY (1998).
3. Cameron, A., *Principles of Lubrication* Longmans (1964).
4. Sommerfeld, A., *Zeits. f. Math. U. Phys.* **40** (1904) 97–155.
5. Steiber, Das-Schwimmlager, *VDI* (1933).
6. Gohar, R., *Elastohydrodynamics*, 2nd ed., Imperial College Press (2001).

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