Chapter 1

Basic Concepts in Mathematical Finance

In this chapter, we give an overview of basic concepts in mathematical finance theory, and then explain those concepts in very simple cases, namely in the single-term finite market.

1.1 Price Processes

Price processes of financial assets are usually modeled as stochastic processes. So mathematical finance theory is based on probability theory, particularly on the theory of stochastic processes.

The price process of an underlying asset is generally denoted by S_t in this book. The process S_t is usually assumed to be positive and is expressed in the following form

$$S_t = e^{Z_t}. \tag{1.1}$$

The time parameter t usually runs in $[0,T]$, $T > 0$, or $[0,\infty)$, and sometimes we consider the discrete time case $(t = 0, 1, \ldots, T)$.

1.2 No-arbitrage and Martingale Measures

In mathematical finance theory, properties of the market where the financial assets are traded are vitally important. If the market works well, then the economy should work well, but if the market does not work well, then the economy shouldn't work.

An important property of the market is its efficiency. This is the "no-arbitrage" or "no free lunch" assumption in mathematical finance theory. A brief definition of arbitrage is: "an arbitrage opportunity is the possibility to make a profit in a financial market without risk and without net investment of capital" (see Delbaen and Schachermayer [30] p.4). The no-arbitrage assumption means that a market does not allow any arbitrage opportunity.

The theory which is constructed on the no-arbitrage assumption is called "arbitrage theory". In arbitrage theory, the martingale measure plays an essential role.

1.3 Complete and Incomplete Markets

If the market has enough commodities, then a new commodity should be a replica of old ones, and we don't need other new commodities. This concept of sufficient commodities in the market is the meaning of completeness in the market.

1.4 Fundamental Theorems

The two concepts introduced above are characterized by the concept of the martingale measure. The following two theorems are well known. (See Delbaen and Schachermayer (2006) [30] or Björk (2004) [9] for details.)

Theorem 1.1. (First Fundamental Theorem in Mathematical Finance). A necessary and sufficient condition for the absence of arbitrage opportunities is the existence of the martingale measure of the underlying asset process.

Theorem 1.2. (Second Fundamental Theorem in Mathematical Finance). Assume the absence of arbitrage opportunities. Then a necessary and sufficient condition for the completeness of the market is the uniqueness of the martingale measure.

If the market is arbitrage-free and complete, then the price of a contingent claim B, $\pi(B)$, is determined by

$$\pi(B) = E_Q[e^{-rT}B], \tag{1.2}$$

where Q is the unique martingale measure and r is the interest rate of the bond. In the case where the market satisfies the no-arbitrage assumption but does not satisfy the completeness assumption, then the price $\pi(B)$ is supposed to be in the following interval:

$$\pi(B) \in \left[\inf_{Q \in \mathcal{M}} E_Q[e^{-rT}B], \ \sup_{Q \in \mathcal{M}} E_Q[e^{-rT}B] \right], \tag{1.3}$$

where $\mathcal{M}$ is the set of all equivalent martingale measures. (See Theorem 2.4.1 in Delbaen and Schachermayer [30].)

1.5 The Black–Scholes Model

The most popular and fundamental model in mathematical finance is the Black–Scholes model (geometric Brownian motion model). The explicit form of the underlying asset process of this model is given by

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}, \tag{1.4}$$

or equivalently in the stochastic differential equation (SDE) form

$$dS_t = S_t \left(\mu dt + \sigma dW_t \right), \tag{1.5}$$

where μ is a real number, σ is a positive real number, and W_t is a Wiener process (standard Brownian motion).

The risk-neutral measure (= martingale measure) Q is uniquely determined by Girsanov's theorem. Under Q the process $\tilde{W}_t = W_t + (\mu - r)\sigma^{-1}t$ is a Wiener process and the price process S_t is expressed in the form

$$S_t = S_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma \tilde{W}_t} \quad \text{or} \quad dS_t = S_t \left(rdt + \sigma d\tilde{W}_t \right), \tag{1.6}$$

where r is the constant interest rate of a risk-free asset. The price of an option X is given by $e^{-rT}E_Q[X]$. The theoretical Black–Scholes price of the European call option, $C(S_0, K, T)$, with the strike price K and the fixed maturity T is given by the following formula:

$$C_K = C(S_0, K, T) = e^{-rT}E_Q[(S_T - K)^+] = S_0 N(d_1) - e^{-rT}KN(d_2), \tag{1.7}$$

where $N(d)$ is the normal distribution function and

$$d_1 = \frac{\log \frac{S_0}{K} + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, \quad d_2 = \frac{\log \frac{S_0}{K} + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}. \tag{1.8}$$

1.6 Properties of the Black–Scholes Model

We summarize the basic properties of the Black–Scholes model as follows.

1.6.1 *Distribution of log returns*

The log return is the increment of the logarithm of S_t,

$$\triangle \log S_t = \log S_{t+\triangle t} - \log S_t = (\mu - \frac{1}{2}\sigma^2)\triangle t + \sigma \triangle W_t, \qquad (1.9)$$

and the log return process is $(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t$.

The distribution of the log return (or the log return process) of the Black–Scholes model is normal. This is convenient for the calculation of the option prices. For example, we have obtained the explicit formula of the price of European call options. However, it is said that the distributions of the log returns in the real market usually have a fat tail and asymmetry. These facts suggest the necessity of considering another model.

1.6.2 *Historical volatility and implied volatility*

Under the setting of the Black–Scholes model, the historical volatility of the process is defined as the estimated value of σ based on the sequential data of the price process S_t. We denote it by $\widehat{\sigma}$. On the other hand, the implied volatility is defined in the following way. Suppose that the market price of the European call option with the strike K, say $C_K^{(m)}$ were given, then the value of σ which satisfies the equation

$$S_0 N(d_1) - e^{-rT} K N(d_2) = C_K^{(m)} \qquad (1.10)$$

is the implied volatility, and this value is denoted by $\sigma_K^{(im)}$. It should be noted that the implied volatility $\sigma_K^{(im)}$ depends on the strike value K but, on the contrary, that the historical volatility $\widehat{\sigma}$ does not depend on K.

We first consider the case where the market value of options obeys the Black–Scholes model, and so the market price $C_K^{(m)}$ is equal to the theoretical Black–Scholes price C_K. In this case the solution of the equation for the implied volatility is equal to the original σ and it holds true that $\sigma_K^{(im)} = \sigma =$ constant. This means that if the market exactly obeys the Black–Scholes model, then the implied volatility $\sigma_K^{(im)}$ should be equal to the historical volatility $\widehat{\sigma}$.

But in the real world this is not true. It is well known that the implied volatility is not equal to the historical volatility, and the implied volatility

$\sigma_K^{(im)}$ is sometimes a convex function of K, and sometimes a combination of a convex part and a concave part. These properties are the so-called "volatility smile or smirk" properties.

1.7 Generalization of the Black–Scholes Model

The Black–Scholes model is a complete market model, but it is said that the real market is incomplete in general. So we have to construct a suitable model for the incomplete market.

1.7.1 *Geometric Lévy process models*

We start from the explicit form of geometric Brownian motion:

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}. \tag{1.11}$$

It is a natural idea to replace the Wiener process with a more general Lévy process Z_t and consider the process

$$S_t = S_0 e^{Z_t}. \tag{1.12}$$

The processes of this type are called the geometric Lévy processes (GLP) or exponential Lévy processes. The [GLP & MEMM] pricing models, which are explained in Chapter 7, are of this type of generalization of Black–Scholes model.

The class of Lévy processes is very diverse and the distributions of S_t may have a at tail and may be asymmetric, and the geometric Lévy process models are generally incomplete market models. These models are studied in Chapter 2.

1.7.2 *Stochastic volatility models*

We start from the SDE form of the Black–Scholes model,

$$dS_t = S_t \left(\mu dt + \sigma dW_t\right). \tag{1.13}$$

When we randomize the volatility σ and consider the equation

$$dS_t = S_t \left(\mu dt + \tilde{\sigma}_t dW_t\right), \tag{1.14}$$

where $\tilde{\sigma}_t$ is a stochastic process, then we obtain the so-called stochastic volatility model.

This model is a very natural one when we think the volatility is dependent on the economic environment. (See Chapter 15 of Cont and Tankov (2004) [25] for this model.)

Notes

For a general introduction to mathematical finance theory, see the following books:

Björk, T. (2004) [9],
Föllmer, H. and Shied, A. (2004) [38],
Jeanblanc, M., Yor, M. and Chesney, M. (2009) [59],
Karatzas, I. and Shreve, S.E. (1998) [64],
Pliska, R.S. (1997) [101],
Shiryaev, A.N. (1999) [116],
Shreve, S.E. (2003) [117],
Shreve, S.E. (2004) [118].

For a study on elementary probability theory there are many books, for example:

Feller, W. (1966) [37],
Williams, D. (1991) [120].