Chapter 2
SPHERICAL BLACK HOLES

In this chapter we look at the vacuum gravitational field produced by a spherical (non-rotating) mass. The spacetime metric for this case, obtained by Karl Schwarzschild in 1916, was the first exact solution of Einstein's equations to be found, although its properties were not fully understood until much later.

We investigate the geometry in the vicinity of the mass M by studying the motion of test particles and light rays. If the space is a vacuum down to the Schwarzschild radius $r = 2GM/c^2$ we obtain the metric of a spherical black hole. In the next chapter we shall study the metric of a rotating (non-spherical) black hole. The surface $r = 2GM/c^2$ constitutes the event horizon of the (spherical) black hole and acts as a one-way membrane for particles and light. We explore the motion of particles and light rays as a spherical mass collapses to form a black hole and what an observer falling into a spherical black hole would experience. To explore the geometry in the vicinity of the event horizon and the maximum extension of the geometry from the external region we introduce various alternative coordinate systems.

2.1 The Schwarzschild metric

The general spherically symmetric metric can be written in the form

$$ds^2 = g_{00}(t,r)c^2dt^2 + g_{11}(t,r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.1}$$

We adopt the timelike convention for the signs of the metric coefficients $(+,-,-,-)$, so $\mathrm{d}s^2 > 0$ is a timelike interval and $\mathrm{d}s^2 < 0$ is a spacelike interval (section 1.3.4). The metric contains two unknown functions of r and t, g_{00} and g_{11}, to be determined from Einstein's field equations.

In the vacuum of space surrounding a massive spherical body Einstein's equations force the functions to be independent of time, and to take the form

$$g_{00} = \left(1 - \frac{2GM}{c^2r}\right), \tag{2.2}$$

$$g_{11} = -\left(1 - \frac{2GM}{c^2r}\right)^{-1}, \tag{2.3}$$

where M is a parameter and G the gravitational constant. Thus the metric becomes

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.4}$$

This is the Schwarzschild metric in Schwarzschild coordinates (t, r, θ, ϕ). It describes the space-time in the exterior vacuum region only. Inside a material body the metric is different and depends on the properties of the body such as the equation of state of the material.

The contravariant components of the metric $g^{\mu\nu}$ satisfy

$$g^{\mu\nu} g_{\nu\rho} = \delta^{\mu}_{\rho},$$

defining $(g^{\mu\nu})$ as the matrix inverse to $(g_{\nu\rho})$. Since the metric is diagonal the matrix inverse is readily found, giving

$$\begin{aligned} g^{00} &= \left(1 - \frac{2GM}{c^2 r}\right)^{-1}, \\ g^{11} &= -\left(1 - \frac{2GM}{c^2 r}\right), \\ g^{22} &= -\frac{1}{r^2}, \\ g^{33} &= -\frac{1}{r^2 \sin^2\theta}. \end{aligned}$$

In relativity the interpretation of the coordinates is obtained from the properties of the metric. We give the interpretation of the coordinates and of the parameter M in the next section.

2.1.1 Coordinates

The notation has been chosen here to be suggestive. The coordinate r does indeed play the role of a radial measure; in fact, the sphere $r =$ constant, $t =$ constant has the (spacelike) metric

$$dl^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.5}$$

from which it is clear that the element of area on the surface is $rd\theta \times r\sin\theta d\phi$, and hence that the area of the surface is $4\pi r^2$. Thus the r coordinate is related to the proper area, that is the area as measured physically by, say, the diminution in intensity of a wave front of light. The angles θ and ϕ are the usual spherical polar angles. Nevertheless, in relativity one should always be aware that a suggestive notation can sometimes be misleading. Later on we shall see that there are regions of space-time (those within $r = 2GM/c^2$) where r is not a spacelike coordinate and t is not timelike. We shall also find that it is sometimes convenient to use other coordinates to describe the Schwarzschild space-time geometry.

2.1.2 Proper distance

The radial increment of proper distance is obtained from the metric by setting $dt = d\theta = d\phi = 0$; it is not dr, but $dr/(1 - 2GM/rc^2)^{1/2}$. The coordinate r therefore does *not* measure proper distance, as would be determined, for example, by a ruler.

2.1.3 Proper time

Similarly, the time coordinate t is *not* the time kept by a standard clock in the exterior gravitational field of the body. We can see this from the metric (2.4). An increment of proper time kept by a stationary clock at a radius r, for which $dr = d\theta = d\phi = 0$, is related to the increment of coordinate time by

$$d\tau = \left(1 - \frac{2GM}{rc^2}\right)^{1/2} dt. \tag{2.6}$$

The increments $\mathrm{d}\tau$ and $\mathrm{d}t$ increasingly agree as $r \to +\infty$. Thus the time coordinate t has the interpretation that it is the proper time kept by a clock at an infinite distance from the body, or a close approximation to that time kept by clocks at large distances.

For the purpose of visualisation we can imagine that the body is surrounded by static shells at each value of the radial coordinate r. A standard clock fixed to a shell measures proper time on that shell, and, of course, standard clocks are identical in that they run at the same rate when they are at the same radius. In addition, we imagine a second clock on each shell that has been adjusted to run at the same rate as the distant clock. This second group of clocks measure coordinate time and run faster than the standard clocks by a factor $(1 - 2GM/rc^2)^{-1/2}$. This slowing down of local clocks relative to a distant clock is the effect of gravitational time dilation. The closer we approach $r = 2GM/c^2$, the faster the coordinate clock runs relative to local proper time, until at $r = 2GM/c^2$ it is running infinitely faster.

The Schwarzschild coordinates (t, r, θ, ϕ) provide a global reference frame that enables us to evaluate quantities in terms of the distance and time measurements of an observer at infinity. As we shall see, the values obtained by a distant observer for time intervals, for energies and for velocities of particles differ from local measurements of these quantities.

2.1.4 Redshift

A consequence of this difference in the measurement of time locally and at infinity is that radiation sent out from a radius r is redshifted when received by a static observer further out. Since the wavelength of radiation is proportional to the period of vibration, equation (2.6) gives us

$$\lambda = \left(1 - \frac{2GM}{rc^2}\right)^{1/2} \lambda_\infty,$$

for the relation between the wavelength λ of radiation emitted at r and the wavelength λ_∞ received at infinity. For the redshift z, defined by

$$z = \frac{\lambda_\infty - \lambda}{\lambda},$$

we get

$$1 + z = \left(1 - \frac{2GM}{rc^2}\right)^{-1/2}.$$

2.1.5 Interpretation of M and geometric units

The quantity M in the Schwarzschild metric arises as a constant of integration in the solution of the Einstein equations. It is in fact the mass of the central gravitating body. This is determined by a comparison of the motion of a body in the Schwarzschild metric with the motion of a body in the Newtonian gravitational field of a mass M, which we shall carry out below. For convenience we define a *geometric mass* m,

$$m = \frac{GM}{c^2}. \tag{2.7}$$

The geometric mass has the dimension of a length, so in SI units, M would be given in kilograms and m in metres.

To simplify further, we replace ct by t and $c\tau$ by τ. The new time is therefore also measured in length units, hence, if we use SI units, in metres. Thus, we have $t(\text{in metres}) = c \times t(\text{in seconds})$, and similarly for τ, but instead of this cumbersome notation we allow the context to distinguish which t and τ we mean. In geometrical units the metric is

$$d\tau^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.8}$$

2.1.6 The Schwarzschild radius

The radial coordinate value $r = 2GM/c^2 = 2m = r_\text{S}$ is called the Schwarzschild radius. We see that something strange appears to happen to the Schwarzschild metric at this value of r, since the metric coefficient g_{11} becomes infinite. For a normal gravitating body the Schwarzschild radius lies within the object where the vacuum Schwarzschild solution ceases to apply. For the Sun, for example, we get $r_\text{S} \sim 3$ km. So, for the dynamics of the Solar System, or even for motion around a neutron star, the problem can be safely ignored. But the question of what happens at the Schwarzschild radius leads to the subject of black holes. Thus, if a spherical mass has radius $R > r_\text{S}$, then the Schwarzschild metric in the region $r > R$ gives its exterior gravitational field; if the region $r \geq r_\text{S}$ is a vacuum then the Schwarzschild metric for $r > r_\text{S}$ gives the exterior gravitational field of a black hole.

Inside $r = r_S$ the Schwarzschild metric is again a solution of Einstein's vacuum field equations and gives the interior geometry of a black hole. As we shall show later, the reason that we cannot follow what happens at the Schwarzschild radius $r = r_S$ in Schwarzschild coordinates is because the coordinates are invalid at this surface, not because of any physical limitation.

At $r = 0$ the coefficient of g_{00} becomes infinite. In contrast to the apparent singularity at the Schwarzschild radius, this singularity cannot be removed by a different choice of coordinates. It is a physical singularity where spacetime curvature becomes infinite and timelike worldlines terminate.

2.1.7 The event horizon

As we shall see, of the many interesting properties of the surface $r = r_S$ the most novel is that it acts as a one-way membrane for information: things can fall into the region $r < r_S$ but cannot emerge from it. For this reason it is called an event horizon and the region of space within it is called a black hole.

Problem 9 *Show that the (proper) circumference of a black hole of mass* m *is* $4\pi m$ *and that the (proper) area is* $16\pi m^2$.

2.1.8 Birkoff's theorem

The coefficients of the Schwarzschild metric do not depend on t; such a metric is said to be *stationary*. Furthermore, there are no cross term involving spacelike and timelike increments (such as $\mathrm{d}t\mathrm{d}\phi$) in the metric. A stationary metric satisfying this further condition is said to be *static*. We have seen that the Einstein equations, which led from the general form (2.1) to the particular form (2.4) or (2.8), imply that a spherically symmetric vacuum solution is static. This result is known as Birkoff's Theorem. It means that if the Sun were to oscillate while remaining spherical the surrounding vacuum would still be described by the Schwarzschild metric (2.4) and therefore that the oscillation could not be detected through the gravitational field. This accords with Newtonian gravity where a spherical distribution of matter gives the same gravitational field as a point of the same mass at its centre. One can emphasize the status of the theorem by rephrasing it as a uniqueness result: the Schwarzschild geometry is the unique spherically symmetric vacuum gravitational field satisfying Einstein's equations.

2.1.9 Israel's theorem

A converse to Birkoff's theorem can also be shown to hold with some additional technical assumptions: a static vacuum gravitational field must be spherical in general relativity, hence must be describable by the metric (2.4). We shall explore this in more detail later. Stated as a uniqueness theorem this becomes: the Schwarzschild geometry is the unique static vacuum gravitational field in general relativity. One is always free to choose different coordinates in relativity, so this result does not say that any spherically symmetric vacuum solution must look like (2.8)! But any such solution can be put into the form (2.8) by a suitable transformation of coordinates.

2.2 Orbits in Newtonian gravity

The simplest way to derive the equations of a test body orbiting in the gravitational field of a spherical mass M in Newtonian theory is to use the conservation of energy and angular momentum. (The energy and angular momentum are constants of the motion.) Since the orbit must lie in a plane (by conservation of angular momentum, or, more directly, by symmetry) we choose the equatorial plane, $\theta = \pi/2$, of a spherical polar coordinate system for convenience.

2.2.1 Newtonian Energy

For a particle of unit mass in orbit in the equatorial plane the conservation of energy gives

$$E_{\mathrm{N}} = \frac{1}{2}\left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2\right) - \frac{GM}{r}, \tag{2.9}$$

where $E_{\mathrm{N}} =$ constant is the total energy per unit mass comprising the kinetic energy of radial and circular motion and the gravitational potential energy.

2.2.2 Angular momentum

Conservation of angular momentum implies (i) that the orbit lies in a plane and (ii) that the magnitude of the angular momentum per unit mass

$$L_{\mathrm{N}} = r^2\frac{d\phi}{dt} \tag{2.10}$$

is constant.

2.2.3 The Newtonian effective potential

Putting together (2.9) and (2.10) we get

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = E_{\mathrm{N}} - \frac{L_{\mathrm{N}}^2}{2r^2} + \frac{GM}{r} = E_{\mathrm{N}} - V_{\mathrm{N}}, \tag{2.11}$$

where V_{N} is called the effective potential. (It is *not* the usual gravitational potential itself because it contains the additional centrifugal term. In fact, it is the potential in a frame rotating with the orbit.)

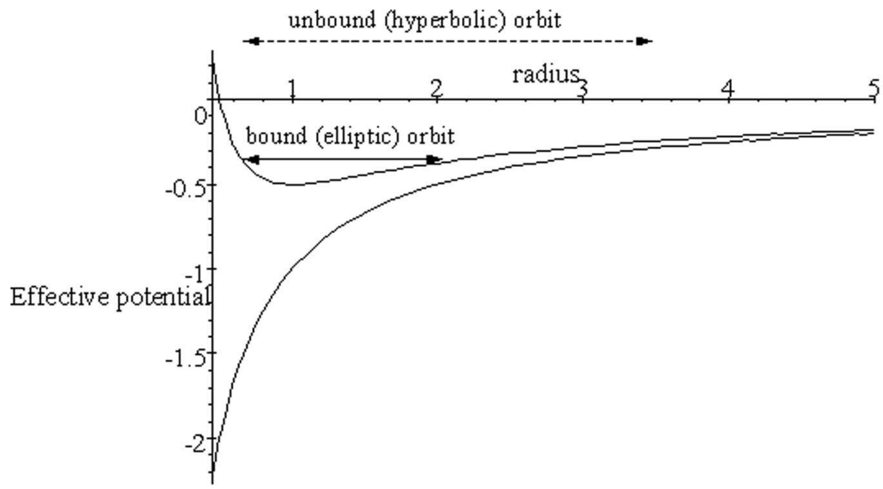

Figure 1 Examples of the Newtonian Effective Potential as a function of radius for zero angular momentum (lower curve) and non-zero angular momentum (upper curve).

2.2.4 Classification of Newtonian orbits

Figure 1 shows a graph of the Newtonian effective potential as a function of radius r in the two cases $L_\mathrm{N} = 0$ and $L_\mathrm{N} = 1$ for a fixed central mass m. In interpreting the figure remember a horizontal line in the figure represents the motion of a particle (having constant total energy) and that the difference between the total energy and a given curve is the kinetic energy as a function of radius for given angular momentum: there is no implication that the motion is radial!

- For $L_\mathrm{N} \neq 0$ we see that $\dot{r}^2$ is positive if $E_\mathrm{N} > V_\mathrm{min}$. Thus the radial velocity is non-zero in these regions.
- From the figure we can see that orbits with $E_\mathrm{N} > 0$ extend to arbitrarily large values of r. These are therefore the unbounded (hyperbolic) orbits.
- Orbits with $E_\mathrm{N} < 0$ (but $> V_\mathrm{min}$) exist between fixed values of r; these are the bound (elliptic) orbits.
- There is one special case when $E_\mathrm{N} = V_\mathrm{min}$ where the value of r is fixed at $r_\mathrm{circ} = L_\mathrm{N}^2/GM$, which is clearly a circular orbit.

Note that, for a central point mass, if $L_\mathrm{N} > 0$ a particle with any positive value of energy will be deflected by the potential back to infinity.

2.3 Particle orbits in the Schwarzschild metric

We are now ready to consider the paths of bodies which are subject to no non-gravitational forces, i.e. the trajectories in spacetime followed by bodies in free fall.

These are given by geodesics of the metric (2.4). There are several ways to obtain such particle paths. The simplest is to make use of the symmetries to derive constants of the motion.

2.3.1 Constants of the motion

Let k^μ be a vector in a direction of symmetry, for example $k^\mu = (1,0,0,0)$ along the t axis. Let $u^\mu = dx^\mu/d\tau$ be the tangent vector to the particle path $x^\mu = x^\mu(\tau)$ in spacetime and consider the component of k^μ along the path, namely $u_\mu k^\mu$. The rate of change of $u_\mu k^\mu$ in the direction u^μ is

$$\begin{aligned} u^\nu (k^\mu u_\mu)_{;\nu} &= u^\nu u_{\mu;\nu} k^\mu + u^\nu u^\mu k_{\mu;\nu} \\ &= u^\nu u_{\mu;\nu} k^\mu + \frac{1}{2} u^\nu u^\mu (k_{\mu;\nu} + k_{\nu;\mu}). \end{aligned}$$

Suppose now that the path is a geodesic. Then $u^\nu u_{\mu;\nu} = 0$ because this is the geodesic equation. Furthermore, it can be shown that $k_{\mu;\nu} + k_{\nu;\mu} = 0$ for a vector along a symmetry direction (see chapter 1). Thus $u^\nu (k^\mu u_\mu)_{;\nu} = 0$ and therefore $k^\mu u_\mu$ is constant along a geodesic (because, in words, the equation says that the derivative of $k^\mu u_\mu$ in the direction of the tangent vanishes).

2.3.2 Conserved Energy

Consider first the time direction, $k^\mu = (1,0,0,0)$. This is fairly obviously a symmetry of the static metric, since the metric coefficients are independent of t. (More formally, the metric is unchanged by the transformation $x^\mu \to x^\mu + \varepsilon k^\mu$.) We therefore have

$$g_{\mu\nu} k^\mu u^\nu = g_{00} k^0 u^0 = g_{00} u^0 = E = \text{constant}. \tag{2.12}$$

because the term in $k^0 = 1$ provides the only non-zero contribution to the sum. Note that we write this expression in terms of u^μ not u_μ because we know that $u^\mu = dx^\mu/d\tau$. Note that $u_0 = g_{0\nu} u^\nu = g_{00} u^0$, so (2.12) is equivalent to

$$u_0 = E \tag{2.13}$$

(or $u_0 = E/c$ in physical units). Hence, from Eq. (2.12), along any geodesic in the Schwarzschild geometry, we have explicitly

$$(1 - 2m/r)\, \frac{dt}{d\tau} = E. \tag{2.14}$$

The tangent vector u^μ to a geodesic is the 4-velocity vector of a particle in free-fall that follows the geodesic. Equivalently, u^μ is the momentum 4-vector of a unit mass particle. To find the meaning of the constant E consider the particle at infinity, then in this limit (2.14) becomes

$$\frac{dt}{d\tau} = E.$$

Now recall from section 1.3.7 that the scalar product $u^\mu_{obs} u_\mu$ is the energy of a particle having 4-velocity u^μ in the frame of a local observer whose 4-velocity is u^μ_{obs}. For a stationary observer at infinity $(u^\mu_{obs}) = (1, 0, 0, 0,)$ so for a particle at infinity

$$u^\mu_{obs} u_\mu = \frac{dt}{d\tau} = E.$$

Thus E is the relativistic energy per unit mass of the particle relative to a stationary observer at infinity. The observer at infinity - sometimes called the bookkeeper - is an important notion. This observer uses the global Schwarzschild coordinates (r, θ, ϕ) and coordinate time t, the time kept by clocks at infinity. For this observer the energy of a free particle is conserved.

2.3.3 Angular momentum

Similarly, for the symmetry in the ϕ direction, we have $k^\mu = (0, 0, 0, 1)$ from which,

$$g_{\mu\nu} k^\mu u^\nu = g_{33} k^3 u^3 = g_{33} u^3 = -L = \text{ constant.} \tag{2.15}$$

Note that $u_3 = g_{3\nu} u^\nu = g_{33} u^3$ so (2.15) is equivalent to

$$u_3 = -L. \tag{2.16}$$

Using $u^3 = d\phi/d\tau$ and the Schwarzschild metric coefficient $g_{33} = -r^2$ this is explicitly

$$r^2 \frac{d\phi}{d\tau} = L = \text{ constant.} \tag{2.17}$$

In this case we interpret conservation of the ϕ component of momentum as the conservation of angular momentum, with L the angular momentum per unit mass relative to an observer at infinity. This interpretation is clearly borne out by comparison with the equivalent Newtonian equation. (Replace proper time τ by Newtonian time t and note that $r \times r\dot{\phi}$ is the angular momentum per unit mass.) This explains the choice of minus sign in (2.16).

Since the gravitational field is spherically symmetric, any free-fall orbit must lie in a plane. (Because all directions are equivalent, there is no way to choose a particular direction out of the plane in which to move. This is not true if the body is acted on by forces, such as rocket motors, because these can define a preferred direction.) As in the Newtonian case we take $\theta = \text{constant} = \pi/2$, the equatorial plane, so $\sin\theta = 1$, which turns out to be the most convenient choice.

Note that we prefer to use energy and angular moment *per unit mass* and hence 4-velocity rather than 4-momentum for particles, since particles of different mass follow the same worldline. We use 4-momentum whenever we want to compare the behaviour of a particle as it tends towards the speed of light with that of a photon.

2.3.4 The effective potential

Our final equation to determine the fourth coordinate along a geodesic comes from the Schwarzschild metric itself. Dividing (2.8) by $d\tau^2$ we get

$$1 = \left(1 - \frac{2m}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - \left(1 - \frac{2m}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\phi}{d\tau}\right)^2. \tag{2.18}$$

These equations, (2.14), (2.17) and (2.18), can be rearranged to give

$$\left(\frac{dr}{d\tau}\right)^2 = E^2 - \left(1 + \frac{L^2}{r^2}\right)\left(1 - \frac{2m}{r}\right) = E^2 - V_{\text{eff}}^2(r). \tag{2.19}$$

The quantity $V_{\text{eff}}(r)$, defined by this equation, is known as the effective potential.

Note that the relation (2.18) (and hence (2.19)) is equivalent to the relation between energy and momentum for a relativistic particle of unit mass, $u_\mu u^\mu = p_\mu p^\mu = 1$, as demonstrated in the following problem.

Problem 10 *Derive equation (2.19) starting from the scalar product of the 4-velocity of the particle* $g^{\mu\nu}u_\mu u_\nu = 1$.

Henceforth we shall denote the mass of a test particle by m_0. This should not be confused with the geometric mass m of a black hole. Bringing together the results of the last sections we see that equations (2.14), (2.17) and (2.19) govern the behaviour of particle orbits in the Schwarschild spacetime. We shall now look at some applications of these equations.

2.3.5 Newtonian approximation to the metric

We are now going to show how the Newtonian equations of motion for a body moving slowly (with respect to light) in an equatorial orbit can be obtained from an approximation to the Schwarzschild metric. Putting $\theta = \pi/2$ and dividing through the Schwarzschild metric by dt^2 we get

$$\left(\frac{d\tau}{dt}\right)^2 = 1 - \frac{2m}{r} - \left(1 - \frac{2m}{r}\right)^{-1}\left(\frac{dr}{dt}\right)^2 - r^2\left(\frac{d\phi}{dt}\right)^2. \tag{2.20}$$

In the Newtonian limit velocities are small, so we can neglect the factor $(1-2m/r)^{-1}$ which multiplies $(dr/dt)^2$. (This factor represents a small correction to an already small quantity.) Note that this is equivalent to ignoring the factor multiplying dr^2 in the metric. We then set $E^2 \approx 1 + 2E_{\text{N}}$, where E_{N} is small. (In physical units this is $E^2 \approx c^2 + 2E_{\text{N}}$.) We also need to use here the relation (2.14) for $d\tau/dt$ which, to the same order of approximation, is

$$\left(\frac{d\tau}{dt}\right)^2 \approx \frac{\left(1 - \frac{2m}{r}\right)^2}{1 + 2E_{\text{N}}} \approx \left(1 - \frac{4m}{r}\right)(1 - 2E_{\text{N}}) \approx 1 - \frac{4m}{r} - 2E_{\text{N}},$$

neglecting products of small terms. With these approximations (2.20) gives the Newtonian equation (2.11). Since $E_{\rm N}$ is the Newtonian energy (per unit mass), E is clearly the relativistic energy (per unit mass) for a slowly moving particle.

We see that the metric

$$d\tau^2 = \left(1 - \frac{2m}{r}\right) dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.21}$$

reproduces the Newtonian equations of motion for slowly moving bodies. We can therefore regard this as the Newtonian approximation to the metric (even though Newtonian gravity is not a metric theory, so does not strictly correspond to a space-time structure with any metric.) We see that in this approximation the gravitational potential produces a redshift, but not a curvature of space. This metric cannot account correctly for the perihelion precession of Mercury or for the bending of light.

Problem 11 *Verify directly from the geodesic equations (1.6) that the Newtonian metric (2.21) reproduces the Newtonian equations of motion for slowly moving bodies. (You will need to compute the components of the affine connection as in chapter 1 and neglect small quantities.)*

2.3.6 Classification of orbits

Figure 2 shows a plot of the effective potential $V_{\rm eff}(r)$ against r. There are still regions of bound and unbound orbits, although there is no reason to believe these will be elliptical or hyperbolic now. Between the two there is a marginally bound (but not parabolic) orbit and the special case of a circular bound orbit. The most striking feature however, is that no amount of angular momentum can keep an orbit of sufficient energy out of the region $r < 2m$. Even worse, for particles with small angular momentum relative to the mass of the central object, in fact if $L < 2\sqrt{3}m$, all orbits end up inside $r = 2m$. Compare this with the Newtonian case (Fig. 1) where any angular momentum prevents a particle reaching $r = 0$. This is the first hint that if we can extend the vacuum field to such small radii then peculiar things will happen.

2.3.7 Radial infall

Let a particle be in radial free fall. Then its 4-velocity is $(u^\alpha) = (u^0, u^1, 0, 0)$ for purely radial motion. From (2.14) we have

$$u^0 = \frac{dt}{d\tau} = \frac{E}{\left(1 - \frac{2m}{r}\right)},$$

and, using $1 = g_{\alpha\beta}p^\alpha p^\beta$, we have

$$u^1 = \pm\left(E^2 - 1 + \frac{2m}{r}\right)^{1/2}. \tag{2.22}$$

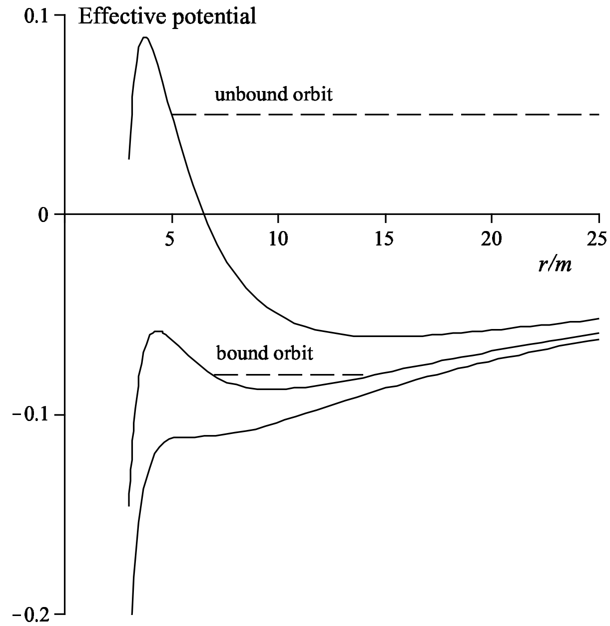

Figure 2 Examples of the relativistic effective potential, plotted as $\frac{1}{2}\left[V_{eff}^2 - 1\right]$ as a function of radius for $L/m = 4.3,\ 3.75,\ 3.46$.

Problem 12 *Obtain the result (2.22).*

The simplest orbit we can treat is the radial infall of a particle released from rest at infinity. In this case the particle at infinity has rest energy only, so $E = 1$. Hence $u^1 = \left(\frac{2m}{r}\right)^{1/2}$.

Problem 13 *Show that the escape velocity from radius r measured by a stationary observer at r is $\left(\frac{2m}{r}\right)^{1/2}$. What is the escape velocity from the horizon at $r = 2m$?*

2.3.8 The locally measured energy of a particle

We have just seen that for a distant observer the total energy per unit mass of the infalling particle has a constant value E. For comparison, the energy $\mathcal{E}$ measured by a local observer maintained at fixed coordinate position r with 4-velocity $(u^\mu) = (dt/d\tau, 0, 0, 0)$ is given by $u^\mu p_\mu$, the projection of the particle 4-momentum along u^μ.

For an observer at rest in the Schwarzschild spacetime, from the metric,

$$d\tau^2 = (1 - 2m/r)dt^2,$$

so

$$(u^\mu) = ([1 - 2m/r]^{-1/2}, 0, 0, 0)$$

and

$$\mathcal{E} = u^\mu p_\mu = \frac{m_0 E}{\left(1 - \frac{2m}{r}\right)^{1/2}}. \tag{2.23}$$

It follows that as measured by local stationary observers the energy of the freely falling particle increases with decreasing r.

How is this compatible with the view of distant observers, who assign a fixed value $m_0 E$ to the energy? Equation (2.23) shows that the two energies are related by a redshift factor. It is helpful to understand why this is just what is required by energy conservation. Suppose the infalling particle is stopped by a stationary observer, its energy $\mathcal{E}$ converted to radiation and sent back to infinity. The energy of this radiation certainly suffers a redshift, so the energy received at infinity is $\mathcal{E}\,(1-2m/r)^{1/2}$. Unless this is equal to the energy $m_0 E$ that we started with, we should be able to construct a perpetual motion machine. Thus, (2.23) exactly expresses energy conservation.

We can obtain one more result from this argument. The kinetic energy of the particle in the local frames of reference is obtained by subtracting the rest mass energy from the total energy:

$$\mathcal{E} - m_0,$$

where as usual we put $c = 1$. Suppose that the particle is brought to rest at r and just the kinetic energy sent back to infinity as radiation. This energy suffers a redshift relative to the distant observer, so the energy received at infinity is

$$(\mathcal{E} - m_0)\left(1 - \frac{2m}{r}\right)^{1/2} = m_0 E - m_0 \left(1 - \frac{2m}{r}\right)^{1/2}.$$

Thus, according to the distant observer, after being brought to rest the particle is left with an energy

$$m_0 \left(1 - \frac{2m}{r}\right)^{1/2}, \tag{2.24}$$

this being the difference between the initial energy at infinity $m_0 E$ and the amount received back in radiation. As $r \to 2m$ the energy sent to infinity is just the original total energy $m_0 E$. The particle cannot give up any more energy to infinity than this. For suppose that its rest mass is converted to radiation at the horizon and sent back to infinity. According to (2.24) the energy received at infinity is redshifted to zero.

2.3.9 Circular orbits

Bringing an infalling particle to rest at a given radius, as in the previous section, is somewhat artificial. More realistically we can consider the circular orbits that are possible about the central mass.

For a circular orbit (r = constant) the two conditions

$$\frac{dr}{d\tau} = 0 \quad \text{and} \quad \frac{d^2 r}{d\tau^2} = 0 \tag{2.25}$$

must be satisfied. The first condition, in conjunction with Eq. (2.19), tells us that $E = V(r)$, so the energy lies on the effective potential curve. The radial equation

(2.19) gives also, in general, by differentiation with respect to τ,

$$\frac{d^2r}{d\tau^2} = -\frac{1}{2}\frac{d}{dr}V_{\text{eff}}^2(r).$$

Hence, the second condition of (2.25) gives

$$\frac{dV_{\text{eff}}^2}{dr} = 2V_{\text{eff}}\frac{dV_{\text{eff}}}{dr} = 0,$$

and hence, finally, $dV_{\text{eff}}/dr = 0$. Therefore circular orbits are possible only at turning points of the effective potential.

Using the explicit form for $V_{\text{eff}}^2(r)$ from (2.19) we find the turning points occur at

$$r = \frac{L^2}{2m} \pm \frac{1}{2}\sqrt{\frac{L^4}{m^2} - 12L^2}. \tag{2.26}$$

For $L^2 > 12m^2$ there are two solutions, the negative sign in (2.26) corresponding to a maximum of $V_{\text{eff}}(r)$, which is an unstable point, and the positive sign to a minimum of $V_{\text{eff}}(r)$, which is therefore stable.

Problem 14 *Show that the solutions obtained by taking the $\pm$ signs in (2.26) correspond respectively to a minimum and a maximum of the effective potential.*

At $L^2 = 12m^2$ there is just one circular orbit: the maximum and minimum of the curve come together at a point of inflection for this value of L. This orbit is marginally stable ($\mathrm{d}^2V_{\text{eff}}/\mathrm{d}r^2 = 0$) and is the innermost (marginally) stable orbit. From (2.26) the condition $L^2 = 12m^2$ gives a radius $r = 6m$ for this innermost stable orbit. Because the orbit is only marginally stable, a particle at $r = 6m$ perturbed inward by a small amount will fall towards $r = 2m$. (At this stage we cannot say what will happen after that.)

2.3.10 Comparison with Newtonian orbits

In Newtonian gravity circular orbits can exist at any radius, however small, and are always bound. The angular momentum per unit mass of a particle in an orbit at r is $L = (mr)^{1/2}$ and its binding energy per unit mass is $E = -m/2r$. A particle moving in from infinity must lose energy before it can go into orbit, and an infinite amount of energy can be extracted from a particle falling into a 'Newtonian' black hole.

For a Schwarzschild black hole stable circular orbits exist for $r \geq 6m$ only, and they are all bound. The angular momentum per unit mass of a particle in a circular orbit at radius r is

$$L = \left(\frac{mr}{1 - 3m/r}\right)^{1/2}$$

and the energy per unit mass is

$$E = \frac{1 - 2m/r}{(1 - 3m/r)^{1/2}}. \tag{2.27}$$

For $r < 6m$ circular orbits are unstable. In the range $4m < r < 6m$ the unstable orbits are bound ($E < 1$). At $r = 4m$ we have $E = 1$ and $L = 4m$. In the range $3m \leq r \leq 4m$ the orbits are unstable and unbound ($E \geq 1$). Thus, a particle coming in from infinity with $E \geq 1$ can (in principle) settle into an unstable circular orbit without losing any energy, provided that it has the appropriate value of angular momentum. Of course, since the orbit is unstable, in practice a small perturbation would eject the particle again. Nevertheless, this is a non-Newtonian feature of the strong gravitational field.

As $r \to 3m$ both E and L tend to infinity, so only zero mass particles (photons) can orbit at this radius.

Problem 15 *Show that the specific angular momentum of a particle in a circular orbit is given by*

$$L = \left(\frac{mr}{1 - 3m/r}\right)^{1/2}.$$

Show that the energy of this particle is

$$E = \left(1 - \frac{2m}{r}\right)\left(1 - \frac{3m}{r}\right)^{-1/2}.$$

Problem 16 *Show that the proper period of the orbit is*

$$\tau = 2\pi\left(\frac{r^3}{m}\right)^{1/2}\left(1 - \frac{3m}{r}\right)^{1/2}$$

and that the coordinate period is

$$T = 2\pi\left(\frac{r^3}{m}\right)^{1/2}.$$

Problem 17 *Show that there exists an unstable circular orbit into which a particle coming from infinity with specific energy $E = 1$ and impact parameter $L/E = 4m$ is inserted.*

2.3.11 Orbital velocity in the frame of a hovering observer

What is the orbital speed of a particle as measured by a hovering observer? We can calculate this as follows. The hovering observer has velocity 4-vector

$$(u_H^\mu) = \left((1 - 2m/r)^{-1/2}, 0, 0, 0\right),$$

and the orbiting particle has 4-velocity

$$(u_\mu) = (E, 0, 0, L),$$

where E is, as usual, the conserved energy per unit mass at infinity. Let the energy per unit mass of the particle be γc^2 with respect to the hovering observer. Then, with $c = 1$, the projection of the momentum of the particle on the 4-velocity of the observer gives

$$\gamma = u_\mu u^\mu = \frac{E}{(1 - 2m/r)^{1/2}}.$$

But recall from special relativity that, in terms of the velocity v of the orbiting particle (and with $c = 1$), $\gamma = (1 - v^2)^{-1/2}$. So using (2.27) for E and solving for v we find

$$v = \frac{(m/r)^{1/2}}{(1 - 2m/r)^{1/2}}.$$

Problem 18 *Show that the local orbital velocity in a circular orbit at radius r equals the escape velocity, $v_{\rm esc} = (2m/r)^{1/2}$, when $r = 4m$ and exceeds the escape velocity for $r < 4m$. Hence deduce that a circular orbit at $r < 4m$ is unbound.*

2.3.12 Energy in the last stable orbit

It is of interest to find the energy of a particle in the innermost stable orbit at $r = 6m$, because this will tell us how much energy can be extracted, in principle, from a particle spiralling slowly inwards. For a circular orbit, we have from (2.19)

$$E^2 = \left(1 - \frac{2m}{r}\right)\left(1 + \frac{L^2}{r^2}\right).$$

With $L^2 = 12m^2$ and $r = 6m$ this becomes

$$E^2 = 8/9.$$

Now a particle at rest at infinity has $E = 1$ (again from (2.19)). So the binding energy per unit mass of a particle in the innermost stable orbit is

$$1 - \sqrt{\frac{8}{9}} = 0.0572,$$

or about 5.7 per cent of $m_0 c^2$ for a particle of mass m_0.

This means that matter spiralling inwards in, for example, an accretion disc (chapter 6), will release 5.7 per cent of its rest mass energy before it plunges into the hole from the last stable orbit. For comparison note that nuclear fusion yields about 0.7 per cent of rest mass energy.

2.4 Orbits of light rays

The behaviour of light can also be used to show the effect of the strong gravitational field at small radii. To obtain the trajectories of light rays consider again the equations of motion for a particle of mass m_0. For conservation of energy we have from (2.14)

$$m_0\,(1 - 2m/r)\,\frac{dt}{d\tau} = m_0 E.$$

For a photon we let $d\tau \to 0$ and $m_0 \to 0$ with $d\lambda = d\tau/m_0$ finite. Then for a ray of light

$$(1 - 2m/r)\,\frac{dt}{d\lambda} = E_{ph},$$

where $m_0 E \to E_{ph}$ as $m_0 \to 0$ and $E \to \infty$. Similarly, from Eq. (2.17), conservation of angular momentum gives

$$r^2\frac{d\phi}{d\lambda} = L_{ph}.$$

where $m_0 L \to L_{ph}$. In addition, we now have

$$d\tau^2 = 0,$$

or, equivalently $g^{\mu\nu}p_\mu p_\nu = 0$, from which

$$\left(\frac{dr}{d\lambda}\right)^2 = E_{ph}^2 - \frac{L_{ph}^2}{r^2}\left(1 - \frac{2m}{r}\right) = E_{ph}^2 - V_{\rm ph}^2, \tag{2.28}$$

where λ is an affine parameter along a ray. The corresponding effective potential $V_{\rm ph}(r)$ has a maximum at $r = 3m$, which is independent of L_{ph}. This is therefore a circular orbit for photons, but because $V_{\rm ph}$ is a maximum the orbit is unstable. The potential $V_{\rm ph}(r)$ has no minima, so there are no stable circular orbits for photons. In other words, although in principle the gravity of a spherical mass can act on light to keep it moving in a circle, in practice this cannot happen.

Drawing together the above results, the equations (2.14), (2.17) and (2.28) govern the behaviour of light rays in the Schwarzschild geometry.

Problem 19 *Evaluate $d\tau/m_0$ in an inertial frame and show that in the limit as $d\tau \to 0$ and $m_0 \to 0$, $d\tau/m_0 \to dt/E_{ph}$.*

Problem 20 *The local speed of light in relativity is always c. Verify that this is true for the Schwarzschild metric. (Hint: obtain an expression for the local velocity of a finite mass particle and consider the limit as its rest mass tends to 0.)*

2.4.1 Radial propagation of light

The 4-momentum of a light ray propagating radially is $(p^\alpha) = (dt/d\lambda, \pm dr/d\lambda, 0, 0)$. Thus, from (2.4)

$$p^0 = \frac{E_{ph}}{(1 - \frac{2m}{r})}$$

where E_{ph} is the conserved photon energy in the Schwarzschild frame. Now from $g_{\alpha\beta}p^\alpha p^\beta = 0$, we obtain $p^1 = \pm E_{ph}$. So

$$(p^\alpha) = \left(\frac{E_{ph}}{(1 - \frac{2m}{r})}, \pm E_{ph}, 0, 0 \right).$$

We have seen that in section 2.1.4 that light of wavelength λ sent from a stationary source at coordinate r to an observer at infinity suffers a redshift

$$\lambda_\infty = \lambda \left(1 - \frac{2m}{r} \right)^{-\frac{1}{2}}.$$

Now we consider the case of a probe in radial free-fall transmitting signals back to a spaceship at infinity. We want to find the relationship between the received and transmitted wavelengths. This can be done by evaluating the scalar product $u^\alpha p_\alpha$, where u^α is the 4-velocity of the probe and p^α is the 4-momentum of a photon. Evaluating the scalar product in a locally freely falling frame in which the probe is stationary gives

$$u^\alpha p_\alpha = h\nu,$$

where ν is the frequency of the signal in the rest frame of the probe. Now we evaluate the scalar product in the Schwarzschild frame, which is the frame of an observer at infinity. The components of the 4-velocity of the probe in the Schwarzschild frame, assuming it has dropped from rest at infinity, are

$$(u^\alpha) = \left(\left(1 - \frac{2m}{r} \right)^{-1}, - \left(\frac{2m}{r} \right)^{\frac{1}{2}}, 0, 0 \right),$$

where u^0 is given by equation (2.14) with $E = 1$. The u^1 component is obtained from $g_{\mu\nu}u^\mu u^\nu = 1$ with the assumption that the probe falls from rest at infinity. The covariant components of the outwardly propagating photon are

$$p_\alpha = \left(h\nu_\infty, -\frac{h\nu_\infty}{(1 - \frac{2m}{r})}, 0, 0 \right).$$

So

$$u^\alpha p_\alpha = \frac{h\nu_\infty}{\left(1-\frac{2m}{r}\right)} + \left(\frac{2m}{r}\right)^{\frac{1}{2}} \frac{h\nu_\infty}{\left(1-\frac{2m}{r}\right)}$$
$$= h\nu_\infty \left(1 - \left(\frac{2m}{r}\right)^{\frac{1}{2}}\right)^{-1} = h\nu,$$

where the second equality follows from the invariance of the scalar product. So finally we get for the wavelength at infinity

$$\lambda_\infty = \lambda \left(1 - \left(\frac{2m}{r}\right)^{1/2}\right)^{-1}. \tag{2.29}$$

Problem 21 *A spaceship plunges from rest at infinity into a black hole. Show that the frequency at which signals sent from infinity with frequency ν_∞ are received by the spaceship at the horizon is $\nu_\infty/2$. What is the frequency of this signal seen by the crew of the spaceship from inside the horizon (assuming that they survive for long enough)?*

2.4.2 Capture cross-section for light

The bending of light by gravity means that non-radial light rays can be captured by a black hole. Later we shall discuss the interior spacetime of a black hole $r < 2m$. For this we shall need to know which light rays can enter this region, that is, we shall need the capture cross-section of the black hole region for light rays. To look at the general properties of the orbits of light rays we return to the corresponding effective potential (Fig. 3)

$$V_{\text{eff}}^2 = \frac{L_{ph}^2}{r^2}\left(1 - \frac{2m}{r}\right).$$

The height of the maximum at $r = 3m$ is $L_{ph}^2/27m^2 = V_{\max}^2$. For $E_{ph} > V_{\max}$ an incoming photon enters $r = 2m$ so it is captured by the hole. For $E_{ph} < V_{\max}$ an incoming photon is scattered by the potential back to infinity, so is not captured. The condition of capture is therefore $L_{ph}/E_{ph} < \sqrt{27}m$.

If the impact parameter of the photon that is just captured by the hole is b, then the capture cross-section of the hole is πb^2. So we have to relate the geometrical quantity b to the dynamical properties of the orbit, L_{ph} and E_{ph}. Now, the angular impact parameter b: $L_{ph} = pb$. For a photon we have $E_{ph} = p$ (recall that $c = 1$ in our units), so we can put $b = L_{ph}/E_{ph}$. This gives us the capture cross-section

$$\sigma = \pi b^2 = 27\pi m^2,$$

whereas the geometrical cross-section is $\pi r_s^2 = 4\pi m^2$.

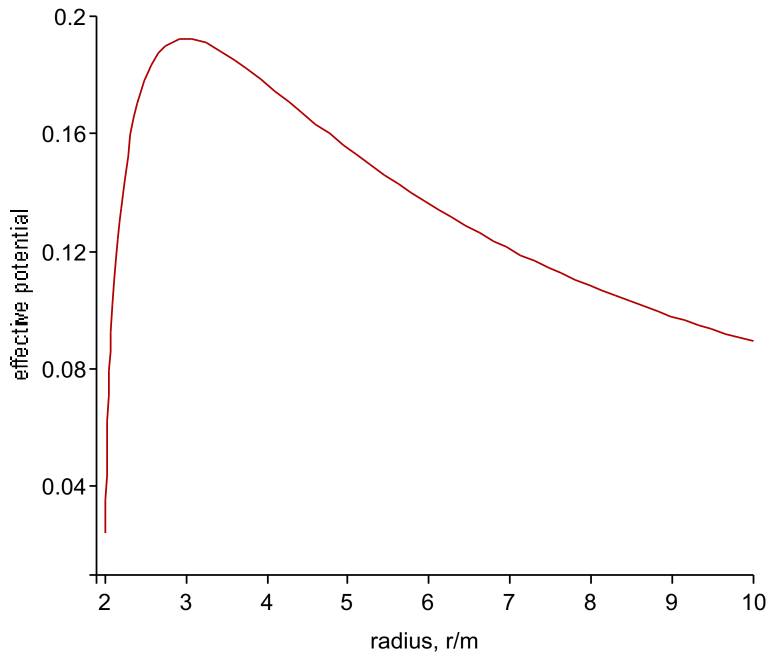

Figure 3 Example of the relativistic effective potential for light rays.

Problem 22 *Using equation (2.28) and the conditions for a circular orbit (section 2.3.9) show that a light ray coming in from infinity with impact parameter* $b = 3\sqrt{3}m$ *enters an unstable circular orbit of radius* $r = 3m$*.*

It follows from problem 22 that a light ray coming from the unstable circular orbit at $r = 3m$ and travelling outwards will have an impact parameter $3\sqrt{3}m$. Thus to a distant observer the apparent radius of the black hole is $3\sqrt{3}m$.

Problem 23 *The black hole at the centre of our Galaxy has a mass of* $\sim 4 \times 10^6 M\odot$*. What is its apparent angular diameter as seen from the Earth, a distance of 8 kpc from the Galactic centre? This is of interest because it is now possible to resolve structures of the order of 40 micro arc seconds using very long baseline interferometry at wavelengths of the order of 1.3mm. (Doeleman et al., 2008)*

2.4.3 The view of the sky for a stationary observer

Suppose that a stationary observer at radius r looking at the sky in the equatorial plane $\theta = \pi/2$ receives a ray of light making an angle ψ' with the outward radial direction. This ray undergoes a proper radial displacement $dr/\left(1 - \frac{2m}{r}\right)^{1/2}$ and a proper tangential displacement $rd\phi$ for each increment $d\lambda$ of affine parameter. From Fig. 4 the angle ψ' is therefore given by

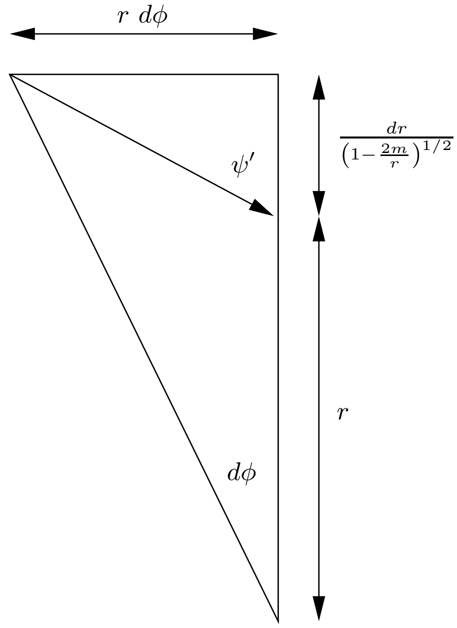

Figure 4 Geometry of a light ray.

$$\sin^2\psi' = \frac{r^2 d\phi^2}{r^2 d\phi^2 + dr^2/\left(1-\frac{2m}{r}\right)} = \frac{\left(1-\frac{2m}{r}\right)}{\left(1-\frac{2m}{r}\right)+\frac{1}{r^2}\left(\frac{dr}{d\phi}\right)^2}.$$

But $dr/d\phi = (dr/d\lambda)/(d\phi/d\lambda) = (dr/d\lambda)/(L_{ph}/r^2)$, and therefore from (2.28)

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{E_{ph}^2}{L_{ph}^2}r^4 - \left(1-\frac{2m}{r}\right)r^2,$$

and finally

$$\sin\psi' = \pm\left(1-\frac{2m}{r}\right)^{1/2}\frac{b}{r}$$

where $b = L_{ph}/E_{ph}$ is the impact parameter. We take the positive square root so that ψ' is near 0 for small b. Then, also,

$$\cos\psi' = \left[1-\left(1-\frac{2m}{r}\right)\frac{b^2}{r^2}\right]^{1/2}. \tag{2.30}$$

Now, according to the calculation of the previous section, an observer close to the horizon at $r = 2m$ will receive only those rays with an impact parameter $b \le \sqrt{27}m$. The corresponding maximum angle is given by

$$\sin\psi'_{\max} \approx \left(1-\frac{2m}{r}\right)^{1/2}\frac{\sqrt{27}}{2}$$

as $r \to 2m$. Thus the black hole fills the whole sky except for a disc of angular diameter $\sqrt{27}\left(1-\frac{2m}{r}\right)^{1/2}$, which closes up to zero for the observer at the horizon.

2.5 Classical tests

We have considered the special cases of radial and circular orbits of particles; now we look at more general orbits. The orbit equations (2.17) and (2.19) allow us to find $dr/d\phi$ as a function of r and hence to compute the paths of bound particles in space. The results can be compared with observation, for example with the perihelion precession of the planet Mercury. Similarly the equations of light rays allow calculation of the bending of light by, for example, the Sun. In these and other ways relativity can be compared with observation. In all of these tests the general relativistic result is found to be at the centre of ever-narrowing error bars. This is discussed further in standard texts on relativity.

These solar system tests involve weak gravitational fields so do not directly imply that strong field effects are correctly predicted by relativity. One example of a strong field test is the speeding up in the period of the binary pulsar (Taylor 1994). But we also look for direct or indirect observation of black holes, having the predicted properties, to verify the theory. (See chapter 5.).

Problem 24 *Obtain the equation of the orbit of a test mass with specific angular momentum L in the equatorial plane of a spherical mass m in the weak field limit $GM/Rc^2 << 1$*

$$\frac{d^2u}{d\phi^2}+u=\frac{m}{L^2}+3mu^2,$$

where $u=1/r$. By seeking an approximate solution of the form $u=l^{-1}(1+\varepsilon\cos\lambda\phi)$ for small eccentricity ε, show that the perihelion rotates by $6\pi m/l$ per orbit.

Obtain similarly the equation of motion of a light ray

$$\frac{d^2u}{d\phi^2}+u=3mu^2.$$

Show that in the absence of the non-Newtonian term on the right of this equation the motion of light is a straight line. With this term, $3mu^2$, included, verify that there is a unique circular orbit for photons.

2.6 Falling into a black hole

From an astrophysical point of view we are interested in the collapse of a sphere of matter that cannot support itself against gravity. This was first discussed by Oppenheimer and Snyder (1939). From Birkoff's theorem on the uniqueness of the Schwarzschild metric we already know that the collapse of a spherical star from the point of view of an external observer is described by the Schwarzschild geometry.

To be precise let us assume that the pressure support in the star has been removed completely and that the surface of the star is in free fall. This gives us a simple way of discussing the approach of the surface towards $r = 2m$. Specifically, we want to know how much time it takes, according to an observer located at a large fixed value of r, for the surface to approach $r = 2m$.

2.6.1 Free-fall time for a distant observer

For radial free fall we can refer to section 2.3.7 to obtain (2.31) below, or we start again from (2.19),

$$\left(\frac{dr}{d\tau}\right)^2 = E^2 - \left(1 - \frac{2m}{r}\right),$$

where we have put $L = 0$ for radial infall. Assuming that the collapse started with a negligible inward speed,

$$\frac{dr}{d\tau} \to 0 \text{ as } r \to \infty,$$

so $E^2 = 1$. Thus

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{2m}{r}. \tag{2.31}$$

However, viewing the collapse from a large distance, we do not record the proper time of the infalling matter, but we use our own proper time, which is the coordinate time t. Thus we use

$$\frac{dt}{d\tau} = \frac{1}{1 - \frac{2m}{r}},$$

from which

$$\frac{dr}{dt} = -\left(\frac{2m}{r}\right)^{1/2}\left(1 - \frac{2m}{r}\right) \tag{2.32}$$

describes the infall of the surface as seen from a distance. We integrate this from a large value of r, $r = R$, say, to close to $r = 2m$ (using r' as the integration variable to distinguish it from the infall radius):

$$t = -\int_R^r \left(\frac{r'}{2m}\right)^{1/2} \frac{r'dr'}{r' - 2m}.$$

To approximate the integral note that the major contribution is from close to $r' = 2m$ so we put $r' = 2m + \varepsilon$ and hence, expanding in powers of ε,

$$\left(\frac{r'}{2m}\right)^{1/2} \frac{r'}{r' - 2m} = \frac{2m}{\varepsilon} + \ldots,$$

where the higher order terms not shown are small compared to $1/\varepsilon$. Hence,

$$t \sim -2m \int_R^r \frac{dr'}{r' - 2m},$$

approximately, and

$$t \sim -2m \log\left(\frac{r}{2m} - 1\right),$$

the contribution from the lower limit $r' = R$, which we have neglected, being much smaller than the term we retain provided that r is close enough to $2m$. This is the time taken for the collapsing surface to approach $r = 2m$ from a large radius. Notice that it tends to infinity as $r \to 2m$.

2.6.2 Light-travel time

But the observer will not see the approach until a photon from the surface has had time to reach him. Therefore, we have to add the time taken by a ray of light to reach a large distance from close to $r = 2m$.

For a light ray $d\tau^2 = 0$ so, from the metric with $d\theta = d\phi = 0$ also

$$\frac{dt}{dr} = \frac{1}{1 - \frac{2m}{r}}.$$

Integrating

$$t' = \int_r^R \frac{r' dr'}{r' - 2m} \sim -2m \log\left(\frac{r}{2m} - 1\right),$$

again taking the main contribution to the integral to come from close to $r = 2m$.

The observer sees the surface at time $T = t + t'$ where, therefore,

$$T = -4m \log\left(\frac{r}{2m} - 1\right). \tag{2.33}$$

This tends to infinity at $r = 2m$. Thus, for an external observer the infall to $r = 2m$ takes an infinite time.

2.6.3 What the external observer sees

The wavelength λ_∞ of radiation received at large r coming from a freely falling source is

$$\lambda_\infty = \lambda \left(1 - \left(\frac{2m}{r}\right)^{\frac{1}{2}}\right)^{-1}.$$

See section 2.4.1 So the red shift is

$$1 + z = \frac{\lambda_\infty}{\lambda} = \left(1 - \left(\frac{2m}{r}\right)^{\frac{1}{2}}\right)^{-1} = \left(\frac{2m}{r}\right)^{-\frac{1}{2}} \left(\left(\frac{r}{2m}\right)^{\frac{1}{2}} - 1\right)^{-1}.$$

To express this in terms of the time T of a distant observer rewrite equation (2.33) as follows:

$$\left(\left(\frac{r}{2m}\right)^{\frac{1}{2}} + 1\right)\left(\left(\frac{r}{2m}\right)^{\frac{1}{2}} - 1\right) = exp\left(-\frac{T}{4m}\right).$$

Hence

$$1 + z = \left(\frac{2m}{r}\right)^{-\frac{1}{2}} \left(\left(\frac{r}{2m}\right)^{\frac{1}{2}} + 1\right) exp\left(\frac{T}{4m}\right).$$

So as $r \to 2m$ the external observer sees the redshift tend to infinity exponentially quickly as the surface of the star approaches $r = 2m$.

Similarly the luminosity decreases proportionally to $1/(1+z)^2$ (one factor of $1+z$ coming from the redshift and one from time dilation: the photon energy falls and so does the rate of arrival of photons). Thus an external observer sees the luminosity of the star decline in time as $\exp(-T/2m)$ as it collapses to $r = 2m$. (A more accurate calculation would take into account non-radial rays leading to a luminosity that falls approximately as $\exp(-T/3\sqrt{3}m)$.)

Seen by a distant observer the velocity dr/dt of an infalling object increases during the initial stages of the fall but later, as it approaches the horizon, it decreases to hover forever just outside the horizon. This assumes the object is viewed through infinitely sensitive instruments that can continue to detect it despite its exponentially fading luminosity. Thus, from the point of view of an external observer a collapsing star would never form a black hole. This caused a lot of confusion in the 1960s and led Russian astronomers to call black holes 'frozen stars'. That black holes do form can be seen by looking at the collapse in the reference frame of the falling object.

2.6.4 An infalling observer's time

From equation (2.31) in terms of the proper time of an infalling observer we have

$$\tau = -\int_R^{2m} \left(\frac{r}{2m}\right)^{1/2} dr = \frac{4}{3}m\left(\left(\frac{R}{2m}\right)^{3/2} - 1\right)$$

as the time to fall from $r = R$ to $r = 2m$. Thus the infalling observer reaches $r = 2m$ from a finite distance in a finite proper time.

A falling object can also be viewed by observers situated on static shells that it passes on its way. The velocity of the object relative to the shell at coordinate r is given by

$$v_{\text{loc}} = \frac{\text{proper distance}}{\text{proper time}} = \frac{dr}{(1 - 2m/r)\,dt}.$$

Using (2.32) for an object falling from rest, we get

$$v_{\text{loc}} = \left(\frac{2m}{r}\right)^{1/2}. \tag{2.34}$$

(An equivalent result was obtained in section 2.3.7.) Thus the locally measured velocity of infall increases steadily with decreasing r and in the limit as $r \to 2m$ we get $v_{\text{loc}} \to 1$ (i.e. in physical units the locally measured speed tends to c).

Problem 25 *a) A particle falls from rest at infinity. Show that the time it takes to fall from $r = 2m$ to $r = 0$ is $4m/3$.*
*b) Show that the maximum time that a particle can take in going from $r = 2m$ to $r = 0$ is $m\pi$. (*Hint: use the metric and the condition that the world line of the falling particle must be light-like in the limit.*)*
c) Show that it corresponds to a particle falling from rest at $r = 2m$.
d) Why would you expect this world line of maximum proper time to be a geodesic?

2.6.5 What the infalling observer feels

Seen from a local frame fixed with respect to the Schwarzschild coordinates a radially infalling particle having $E = 1$ falls with speed $v_{\text{loc}} = (2m/r)^{1/2}$ (section 2.31, or Eq. (2.34)). If we work from the frame of the freely falling particle, the speed of the coordinate lattice in this frame is also $v_{\text{loc}} = (2m/r)^{1/2}$. Let τ be the proper time of the particle. The acceleration of the stationary frame is $dv_{\text{loc}}/d\tau = (dv_{\text{loc}}/dr)(dr/d\tau) = (dv_{\text{loc}}/dr)v_{\text{loc}} = m/r^2$.

As in Newtonian physics the dependence of the acceleration on radial distance means that an extended body will feel a tidal force. We want to calculate this tidal force from the point of view of the infalling observer (who will after all be subject to the force). We follow the derivation in Taylor and Wheeler (2000). Any two observers agree on their relative acceleration, so the relative acceleration of the head and the feet of a radially aligned infalling observer is

$$dg = \frac{2m}{r^3}dr.$$

To get the tidal force we need the proper distance corresponding to a coordinate distance dr (because our height that we measure in metres is our proper height in free fall, not, in principle, a coordinate displacement in Schwarzschild coordinates, although the two will turn out to be numerically the same).

In a stationary frame the proper distance corresponding to a displacement dr is $dr\,(1 - 2m/r)^{-1/2}$. For the freely falling observer moving with speed $v_{\text{loc}} = (2m/r)^{1/2}$, this length is contracted by a factor

$$1/\gamma = \left(1 - v_{\text{loc}}^2\right)^{1/2} = (1 - 2m/r)^{1/2}.$$

So the proper length in the freely falling frame is

$$\gamma^{-1}dr\,(1 - 2m/r)^{-1/2} = dr.$$

The tidal acceleration on a body of extension Δr is therefore approximately $(2m/r^3)\Delta r$. At some point this will begin to cause pain for the infalling observer. We want to calculate how long this pain must be endured before the singularity at $r = 0$ is encountered.

Suppose that our criterion for the onset of pain is $\Delta g = g_{\rm E}$, the gravity at the Earth's surface. This will occur at a radius $r_p = (2m\Delta r/g_{\rm E})^{1/3}$. The infall time to $r = 0$ is obtained by integrating (2.31):

$$\tau = -\int_{r_p}^{0} \left(\frac{r}{2m}\right)^{1/2} dr = \frac{2r_p^{3/2}}{3(2m)^{1/2}} = \frac{2}{3}\left(\frac{\Delta r}{g_{\rm E}}\right)^{1/2}.$$

Note that this is independent of the black hole mass.

Problem 26 *Show that for a standard observer of height* $2m$ *the infall time from the threshold of pain to death at the singularity is about* $0.3s$*. Show that for an observer falling into a* $1M_\odot$ *black hole the threshold of pain occurs well outside the Schwarzschild radius. For what mass black hole does the pain threshold coincide with the Schwarzschild radius?*

Problem 27 *What mass of black hole would allow an observer in free fall to survive for a year inside the horizon before being destroyed?*

2.7 Capture by a black hole

In section 2.4.2 we looked at the probability that a ray of light would be captured by a black hole taking into account the bending of light by the gravitational field of the black hole. It is of interest to calculate the corresponding capture cross-section of a black hole for particles. The method is similar: any particle with energy $E > V_{\rm eff}^{\rm max}$ is captured by the hole. This gives us a critical angular momentum for capture. Instead of angular momentum (for a unit mass particle) L we use the impact parameter b defined in terms of the particle velocity at infinity v by $L = vb$. The capture cross-section is πb^2. With a lot of algebra it can be shown that the maximum of

$$V_{\rm eff}^2 = \left(1 - \frac{2m}{r}\right)\left(1 + \frac{L^2}{r^2}\right)$$

occurs for $r = (L^2/2m)\left[1 - (1 - 12m^2/L^2)^{1/2}\right]$ and is given by

$$(V_{\rm eff}^{\rm max})^2 = \frac{1}{54}\left[l^2 + 36 + (l^2 - 12)\left(1 - \frac{12}{l^2}\right)^{1/2}\right], \qquad (2.35)$$

where $l = L/m$.

For the rest of the calculation we consider two limiting cases.

2.7.1 Case I: Capture of high angular momentum particles

Inserting the condition $l \gg 1$ into (2.35) and using the binomial expansion gives

$$(V_{\rm eff}^{\rm max})^2 \approx \frac{l^2 + 9}{27}.$$

The critical angular momentum below which capture will occur is therefore $l_*^2 = 27E^2 - 9$. Thus, for a particle of energy E per unit mass and momentum p per unit mass, the critical impact parameter is

$$b_* = \frac{L_*}{p} = \frac{ml_*}{(E^2-1)^{1/2}},$$

and the cross-section is therefore

$$\sigma = \pi b_*^2 = \frac{\pi m^2 l_*^2}{(E^2-1)}.$$

For large E ($E \gg 1$) we have $(E^2-1)^{-1} = E^{-2}(1-1/E^2)^{-1} \approx E^{-2}(1+1/E^2)$, so

$$\sigma \approx \frac{\pi m^2}{E^2}\left(1+\frac{1}{E^2}\right)\left(27E^2-9\right) = 27\pi m^2\left(1+\frac{2}{3E^2}\right).$$

Note that as $E \to \infty$, $\sigma \to 27\pi m^2$, and we recover the capture cross-section for a photon.

2.7.2 Case II: Capture of low energy particles

For a particle with low energy ($v \ll 1$) we put $(V_{\text{eff}}^{\text{max}})^2 = E^2 \approx (1+\frac{1}{2}v^2)^2 \approx 1+v^2$ as the limit for capture and solve for the critical angular momentum l_* in terms of v. This gives

$$l^2 \approx 16 + 27v^2 + \ldots,$$

neglecting terms in v^4 and higher. Thus

$$\sigma = \frac{16\pi m^2}{v^2},$$

which agrees with our intuition that a particle moving slowly enough will be captured by the gravity of the black hole.

2.8 Surface gravity of a black hole

In this section we are going to derive an important quantity associated with a black hole, called its surface gravity by analogy with the acceleration due to gravity g of Newtonian theory. First we need to determine the acceleration of a particle at rest with respect to a local observer at rest, and then to determine how this acceleration appears from infinity.

2.8.1 The proper acceleration of a hovering observer

The proper or rest frame acceleration a of a particle is related to its 4-acceleration a^μ by equation (1.13):

$$a^\mu a_\mu = -a^2,$$

where

$$a^\mu = \frac{du^\mu}{d\tau} + \Gamma^\mu_{\rho\sigma} u^\rho u^\sigma.$$

The components of the 4-velocity of a hovering observer are

$$u^\mu = \left(\left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}}, 0, 0, 0 \right).$$

The only non-zero component of the 4-acceleration is

$$a^1 = \frac{du^1}{d\tau} + \Gamma^1_{00} \left(u^0\right)^2$$

where

$$\Gamma^1_{00} = \frac{m}{r^2}\left(1 - \frac{2m}{r}\right).$$

As $u^1 = 0$ we obtain

$$a^1 = \frac{m}{r^2}, \qquad \text{and} \qquad \mathrm{a}_1 = \mathrm{g}_{11}\mathrm{a}^1.$$

Finally ,

$$a^\mu a_\mu = -\left(\frac{m}{r^2}\right)^2 \left(1 - \frac{2m}{r}\right)^{-1},$$

so

$$a = \frac{m}{r^2}\left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}}.$$

Note that the proper acceleration goes to infinity at the horizon, so only a photon can hover at the horizon.

2.8.2 Surface gravity

What is the gravitational acceleration at the event horizon as seen from infinity? The result in fact has the Newtonian form m/r^2. (Note that this is not the locally measured gravitational acceleration at the hole, which we have just calculated in the previous section, so it is not the same as Newtonian gravity, nor is it the Newtonian approximation at large distances that we are dealing with, even though this has the same form.) We shall give two ways of arriving at this answer, offering different levels of physical insight and mathematical sophistication. First a simple physical approach.

Consider a massless inextensible string used by a distant observer to raise a particle of unit mass at a radius r through a distance dl. In this process the local energy of the particle increases by $dE_r = g_r dl$, where g_r is the acceleration due to gravity at r. The energy expended by the distant observer in this process is $dE_\infty = g_\infty dl$. By conservation of energy the two energies are related by a redshift factor. (Otherwise one could make a perpetual motion machine by converting energy to radiation and sending it between the two ends of the string.) Thus

$$\frac{g_r}{g_\infty} = \frac{E_r}{E_\infty} = \left(1 - \frac{2m}{r}\right)^{-1/2} \tag{2.36}$$

so

$$g_\infty = \frac{m}{r^2}. \tag{2.37}$$

Here g_r is the proper acceleration felt by an observer hovering at radius r, which would be interpreted as the acceleration due to gravity at the radius r. It is given by (2.8.1). As $r \to 2m$, $g_r \to \infty$. Nevertheless, the force required at infinity to hold the particle of unit mass hovering at the horizon is $g_\infty = m/(2m)^2 = 1/4m$. We call this the *surface gravity* κ of the black hole. Our result is therefore that the surface gravity of a Schwarzschild black hole is $\kappa = 1/4m$.

2.8.3 Rindler coordinates

To investigate further what the spacetime is like in the vicinity of the event horizon we introduce a new set of coordinates in this region. For the time coordinate we use the Schwarzschild t, but we define a new radial coordinate ξ centred on $r = 2m$ and symmetrical about this origin by

$$r - 2m = \frac{\xi^2}{8m}$$

where the factor of 8 has been inserted with hindsight to tidy up the resulting expressions. We also define

$$\kappa = \frac{1}{4m},$$

which we have seen is the surface gravity of the black hole. Then we find

$$1 - \frac{2m}{r} = \frac{(\kappa\xi)^2}{(1 + \kappa^2\xi^2)} \tag{2.38}$$

$$\sim \kappa^2\xi^2 \tag{2.39}$$

near $\xi = 0$ (i.e. near $r = 2m$). So near the horizon the metric is approximately

$$d\tau^2 \sim \kappa^2\xi^2 dt^2 - d\xi^2 - \frac{1}{4\kappa^2} d\widetilde{\omega}^2, \tag{2.40}$$

where

$$d\widetilde{\omega}^2 = d\theta^2 + \sin^2\theta d\phi^2.$$

The coordinates (t, ξ, θ, ϕ) are called Rindler coordinates.

This metric has a simple interpretation. Consider just the (t, ξ) plane and let

$$\begin{aligned} T &= \xi \sinh \kappa t \\ X &= \xi \cosh \kappa t. \end{aligned} \tag{2.41}$$

Substitution into the (t, ξ) part of the Rindler metric (2.40) shows that

$$d\tau^2 = dT^2 - dX^2, \tag{2.42}$$

which is just Minkowski spacetime in two dimensions.

Problem 28 *Derive equation (2.42).*

So for an observer hovering near the horizon the Rindler metric is just Minkowski spacetime in an unusual coordinate system! Near the horizon of a spherical black hole the geometry is approximately flat.

This may require some explanation. To remain near the horizon keeping r close to $2m$ an observer must apply an acceleration. (Un-accelerated trajectories are radial geodesics which fall into the hole.) It is along this accelerated trajectory that the metric is approximately flat. This is a special feature. In a general spacetime it is only (unaccelerated, freely falling) geodesic observers who see their neighbourhoods as approximately flat.

Problem 29 *The proper time interval given by the Rindler metric in two dimensions is*

$$d\tau^2 = \kappa^2 \xi^2 dt^2 - d\xi^2.$$

On the curve $\xi = 1/a =$ constant we have, from Eq. (2.41),

$$\begin{aligned} T &= a^{-1} \sinh \kappa t, \\ X &= a^{-1} \cosh \kappa t. \end{aligned}$$

From this calculate the components of the proper acceleration on this trajectory in (T, X) coordinates, namely $d^2T/d\tau^2$ and $d^2X/d\tau^2$, and show that the magnitude of the proper acceleration is a.

Thus a is the proper acceleration on the curve $\xi = 1/a$. From Eq. (2.39) $a \sim \kappa \left(1 - \frac{2m}{r}\right)^{-1/2}$. Comparing this with (2.8.1) we see that κ is approximately the acceleration of a particle near the horizon as seen from infinity, a relation that becomes exact at the surface of the hole at $r = 2m$. The quantity κ is therefore equivalent to the Newtonian acceleration due to gravity ("g") which again justifies calling it the surface gravity of the hole.

2.9 Other coordinates

The Schwarzschild metric appears to be a viable metric, and a solution of Einstein's equations, for $r < 2m$. However, in this region, the coordinate r is timelike and the coordinate t spacelike! For example, for a body in the region $r < 2m$ with $\theta =$ constant, $\phi =$ constant, $t =$ constant we have $d\tau^2 = g_{11}dr^2 > 0$ so a (non-zero) dr is a timelike interval in this region. This means that a body in this region cannot remain at constant r (since dr must be non-zero to make the worldline of the body timelike). We shall see that it must fall into the singularity at $r = 0$. Furthermore, since the metric coefficients depend on time, the metric within $r = 2m$ is not static.

However, the connection between the space within $r = 2m$ and outside $r = 2m$ cannot be examined in Schwarzschild coordinates, because the metric is invalid at $r = 2m$. We say that there is a coordinate singularity at $r = 2m$, because the problem turns out to be with the coordinate system, not with the physics. (A similar problem occurs at the pole of a spherical polar coordinate system in Euclidean space, where $\theta = 0$, $0 \leq \phi \leq 2\pi$ is not a circle, but a single point.) We can remedy this by finding one or more alternative sets of coordinates that do not have a coordinate singularity at $r = 2m$. We shall follow the historical development, introducing first the two sets of Eddington–Finkelstein coordinates and then the Kruskal coordinates, which cannot be expressed in elementary functions, but which have the advantage that they cover the whole spacetime.

2.9.1 Null coordinates

Consider first Minkowski spacetime. On an outgoing light ray (r increasing as t increases), $u = t - r$ is a constant. Different values of u label different light rays according to the initial point (the value of r when $t = 0$). As usual for convenience we use units with $c = 1$. Similarly, on ingoing rays the quantity $v = t + r$ is a constant, different values of the constant labeling different rays. Along an ingoing ray u changes, so u labels points on an ingoing ray. Conversely v labels points on a given outgoing ray. Because they label light rays the coordinates u and v are often called null coordinates. We can use either u or v (or both) as an alternative 'time' coordinate. For example, making the transformation

$$t = v - r \tag{2.43}$$

we get

$$d\tau^2 = dv^2 - 2dvdr - r^2 d\widetilde{\omega}^2.$$

Using u and v as coordinates in the (r, t) plane, we have

$$d\tau^2 = dudv - \frac{1}{4}(v - u)^2 d\widetilde{\omega}^2.$$

2.9.2 Eddington–Finkelstein coordinates

To carry out the analogous transformation in Schwarzschild spacetime we need to find the equation of a radial null ray. This is obtained by integrating $d\tau^2 = 0$ for a radial ray. We have from (2.8)

$$dt^2 = \frac{dr^2}{\left(1 - \frac{2m}{r}\right)^2} = dr_*^2, \tag{2.44}$$

where we have defined the new radial coordinate r_* in such a way that radial null rays are $d(t \pm r_*) = 0$ or $t \pm r_* =$ constant. Integrating the second pair of equations in Eq. (2.44) we get

$$r_* = r + 2m \log \left| \frac{r - 2m}{2m} \right|. \tag{2.45}$$

Let

$$v = t + r_*.$$

We are going to use v as a time coordinate, replacing t. Substitution in the Schwarzschild metric gives

$$d\tau^2 = \left(1 - \frac{2m}{r}\right)(dt^2 - dr_*^2) - r^2 d\widetilde{\omega}^2 \tag{2.46}$$

$$= \left(1 - \frac{2m}{r}\right) dv^2 - 2dr dv - r^2 d\widetilde{\omega}^2. \tag{2.47}$$

The final expression is the form of the Schwarzschild metric in ingoing Eddington-Finkelstein coordinates. In this form the metric is non-singular at $r = 2m$, because $\det(g) = -(2m)^4 \sin^2\theta \neq 0$. To plot the spacetime in a (t, r) plane we introduce the new time coordinate corresponding to v and r,

$$t_* = v - r$$

(compare (2.43)). Figure 5 shows the (t_*, r) plane extending from $r = 0$ to $r = \infty$. In particular we can see what happens to the light cones as we approach $r = 2m$, which is a non-singular surface in these coordinates. Either from the figure, or from the metric (2.47), we see that $r = 2m, \theta =$ constant, $\phi =$ constant is a null ray, and that the surface $r = 2m$ is generated by these null rays (for different values of θ and ϕ).

2.10 Inside the black hole

We are now in a position to consider the properties of the black hole inside the event horizon. We can show that within the surface $r = 2m$ all future pointing directions point inwards to decreasing r. This means it is impossible to escape from within the region $r < 2m$, since real bodies can travel only in timelike (or null) directions.

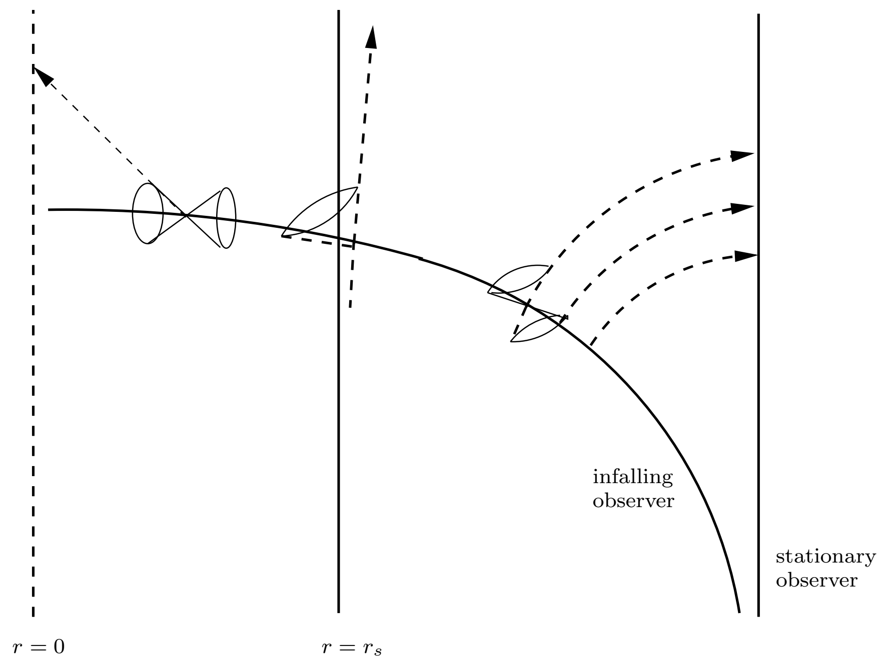

Figure 5 Spacetime diagram in Eddington–Finkelstein coordinates showing an observer falling towards a black hole.

The surface $r = 2m$ therefore acts as a one-way membrane for matter and radiation: nothing can escape into the surrounding spacetime. The surface $r = 2m$ is the event horizon, and the region $r < 2m$ is a 'black hole'.

To demonstrate this take the metric (2.47) and write it as

$$2drdv = -\left[d\tau^2 - \left(1 - \frac{2m}{r}\right) dv^2 + r^2 d\widetilde{\omega}^2\right] \tag{2.48}$$

$$= -\left[d\tau^2 + \left(\frac{2m}{r} - 1\right) dv^2 + r^2 d\widetilde{\omega}^2\right]. \tag{2.49}$$

For a timelike or null displacement $d\tau^2 \geq 0$ so for $r < 2m$ the sum of squares on the right hand side is positive and the overall expression is negative. Recall now that on a radial null ray $dv = d(t + r_*) = d(t_* + r) = 0$, and hence that a displacement dv lies within the future light cone if $dv > 0$. So to make the left hand side negative for a future-pointing displacement we have to have $dr < 0$. Thus a future-pointing displacement is necessarily in an inward direction.

The event horizon cuts off all events within $r = 2m$ from the outside world, so an explorer who crosses the event horizon of a black hole can never return or signal his experiences back to the outside world. A more pressing problem is that the observer cannot avoid destruction at the singularity that lies in their future.

2.10.1 The infalling observer

Our discussion of section 2.4.3 does not apply inside the black hole, because there are no stationary observers there. We can however transform to infalling observers outside the black hole and continue the results for these into the interior. This will enable us to provide a picture of what the infalling observer sees approaching the singularity.

Let the plunging observer fall from rest at infinity and denote his local frame of reference by S. As before, his velocity with respect to a stationary observer is

$$v = \left(\frac{2m}{r}\right)^{1/2}. \tag{2.50}$$

Let the stationary observer's local frame be S′. The angle of the incoming light ray in this frame is ψ'; let the corresponding angle in the S-frame be ψ. The usual special relativistic velocity transformation applies between the local frames, so

$$\cos\psi = \frac{\cos\psi' - v}{1 - v\cos\psi'} \tag{2.51}$$

(recalling that we use $c = 1$). Thus $\cos\psi < \cos\psi'$ so $\psi > \psi'$: the observer in S sees less of the sky as black.

Using now our previous result (2.30) for $\cos\psi'$ we have

$$\cos\psi = \frac{\left[1 - \left(1 - \frac{2m}{r}\right)\frac{b^2}{r^2}\right]^{1/2} - v}{1 - v\left[1 - \left(1 - \frac{2m}{r}\right)\frac{b^2}{r^2}\right]^{1/2}}. \tag{2.52}$$

What happens to (2.52) as $r \to 2m$? Inserting $r = 2m$ directly gives an indeterminate form. So we put $r = 2m + \delta$ and expand both numerator and denominator by the binomial theorem for small δ to get

$$\cos\psi \simeq -\frac{1 - \frac{b^2}{4m^2}}{1 + \frac{b^2}{4m^2}}.$$

The maximum value of b for which a ray can reach the horizon is again $\sqrt{27}m$, so inserting this value of b gives

$$\cos\psi_{\rm max} = -\frac{23}{31}$$

or $\psi_{\rm max} = 138^{\rm o}$. All rays up to this corresponding to smaller values of b can be seen.

Evidently the aberration formula (2.52) applies also inside the horizon. The angle ψ varies smoothly as the horizon is crossed. So let us see what the plunging observer sees at $r = m$ and as $r \to 0$. At $r = m$ (2.52) together with (2.50) and $b = \sqrt{27}m$ gives $\cos\psi_{\rm max} = -0.598$, or $\psi_{\rm max} = 127^{\rm o}$. Thus the observer sees more of the sky as black as he falls in.

As $r \to 0$ (2.52) gives

$$\cos\psi \sim -\left(\frac{r}{2m}\right)^{1/2} \tag{2.53}$$

independent of b. Hence in the limit all rays are seen at $\cos\psi = 0$, or $\psi = \pi/2$. As the observer crashes into the singularity the outside world appears as a thin bright ring.

Problem 30 *Derive (2.53) from equation (2.52).*

2.11 White holes

The Eddington–Finkelstein coordinates we used above were based on inward going null rays that entered $r = 2m$ in the future. It is natural to ask if we can make a similar picture with outgoing rays by taking $u = t - r_*$ as the 'time' coordinate. Straightforward substitution into the Schwarzschild metric gives

$$d\tau^2 = \left(1 - \frac{2m}{r}\right) du^2 + 2drdu - r^2 d\widetilde{\omega}^2.$$

This describes (part of) a spacetime which is the time-reverse of the black hole in which all future-directed null or timelike paths emerge from $r < 2m$ into the surrounding space. This object is called a white hole. Note that both black holes and white holes are mathematical solutions of the vacuum Einstein equations but that does not mean they are both physical possibilities. We shall present arguments later (chapter 6) to support the existence of black holes, but there is no evidence for the existence of white holes.

2.12 Kruskal coordinates

The Eddington–Finkelstein coordinates cover either the part of spacetime representing a black hole region or the part representing a white hole, but not both. It would obviously be convenient to be able to look at both regions in the same picture. This is made possible by yet another coordinate system, the Kruskal coordinates.

Return to the Schwarzschild metric in the form

$$d\tau^2 = \left(1 - \frac{2m}{r}\right)(dt^2 - dr_*^2) - r^2 d\widetilde{\omega}^2$$

and write this in double null coordinates u and v as

$$d\tau^2 = \left(1 - \frac{2m}{r}\right) dudv - r^2 d\widetilde{\omega}^2.$$

Now define

$$\begin{aligned} U &= -\exp(-u/4m) \\ V &= \exp(v/4m). \end{aligned} \tag{2.54}$$

This can be shown to lead to

$$d\tau^2 = \frac{32m^3}{r}\exp(-r/2m)dUdV - r^2 d\widetilde{\omega}^2, \tag{2.55}$$

where r is considered to be a function of U and V through

$$VU = -\exp\left(\frac{v-u}{4m}\right) = -\exp(r_*/2m),$$

since $v - u = 2r_*$. The Kruskal coordinates are (U, V, θ, ϕ). In Kruskal coordinates light rays are represented by lines at 45^o (with $c = 1$).

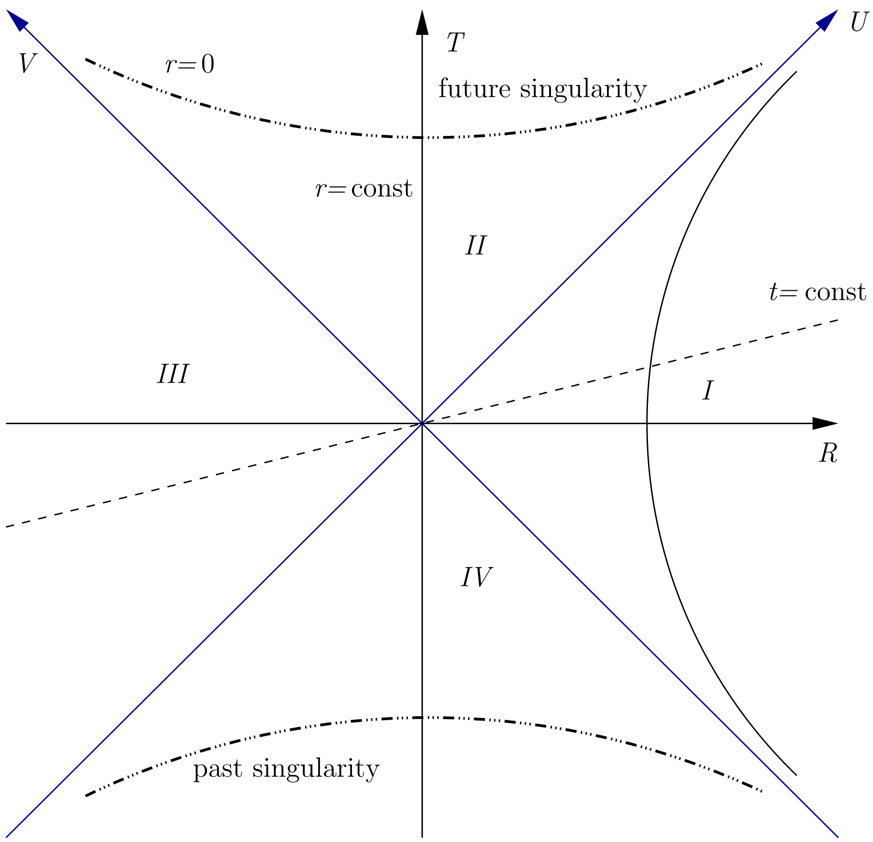

Figure 6 Schwarzschild spacetime in Kruskal (U, V) coordinates.

The metric (2.55) covers four regions as shown in Fig. 6. The two regions $r > 0$ and $r < 0$ are separate asymptotically flat universes. Note the two singularities at $r = 0$, one in the past and one in the future, which appeared separately in the pictures of the two sets of Eddington–Finkelstein coordinates. Figure 7b shows the surface of a collapsing star within which the metric would differ from the Kruskal picture.

There is one aspect of the Kruskal picture that can be misleading. Each point in the plane represents a 2-sphere of that radius at that time. This is not the same as saying that either of the angles can be re-created by rotating the diagram about an axis (which is how one often re-introduces an additional dimension). It is absolutely not the case that one can get from region I to region II, even in principle, by moving on a circle of constant ϕ! (And conversely, of course, rotation in Euclidean space in spherical polar coordinates does not take one to a region of negative r.)

2.12.1 The singularities at $r = 0$ and cosmic censorship

After falling through the event horizon a particle must continue to move to smaller values of r. This can continue in principle until the particle approaches $r = 0$ where the curvature becomes infinite. Any real body will suffer large tidal distortions as it approaches $r = 0$ and will be destroyed. But what happens to our hypothetical point particle? It reaches the edge of spacetime beyond which the theory has nothing to say. This is a true physical singularity, unlike the coordinate singularity at $r = 2m$.

To a certain extent it is a harmless singularity as far as physics in the outside world is concerned. This is because the unpredictability is hidden behind an event horizon and does not affect the black hole exterior. The white hole singularity at $r = 0$ in the past is a different matter. Here any unpredictability emerges into the surrounding spacetime making physics impossible. (Because, for example, any detection of a particle can be attributed to its emergence from the singularity rather than having to have a prior cause within the physical universe.) Thus the white hole singularity is 'naked'.

We shall see further examples of naked singularities later. The cosmic censorship hypothesis (put forward by Penrose) seeks to make naked singularities illegal as far as 'normal' physics is concerned by proposing that naked singularities cannot form from normal matter. There is no proof of this conjecture, but attempts to violate it have only helped to frame it more precisely. For the present, we observe that the formation of a black hole by the collapse of matter is not a time-symmetric process and leads to a black hole without the associated white hole.

2.12.2 The spacetime of a collapsing star

Figure 7 shows how the collapse of a sphere of matter to a singularity at $r = 0$ can be represented in (a) Eddington–Finkelstein coordinates and (b) Kruskal coordinates respectively. Note that within $r = 2m$ the Eddington–Finkelstein space and time coordinates (t, r_*) swap roles so the singularity at $r_* = r = 0$ occurs at a constant 'time' and is therefore spacelike, despite what the picture appears to show at first glance! This is clearly represented in the Kruskal picture. Note also that the white hole region has disappeared to be replaced by a solution of the field equations for the interior of the star. Although we make no attempt to find an interior solution, the pictures are drawn under the (correct) assumption that, prior to the final collapse, the interior will be a regular spacetime with a *decreasing* curvature as we go through the star towards $r = 0$.

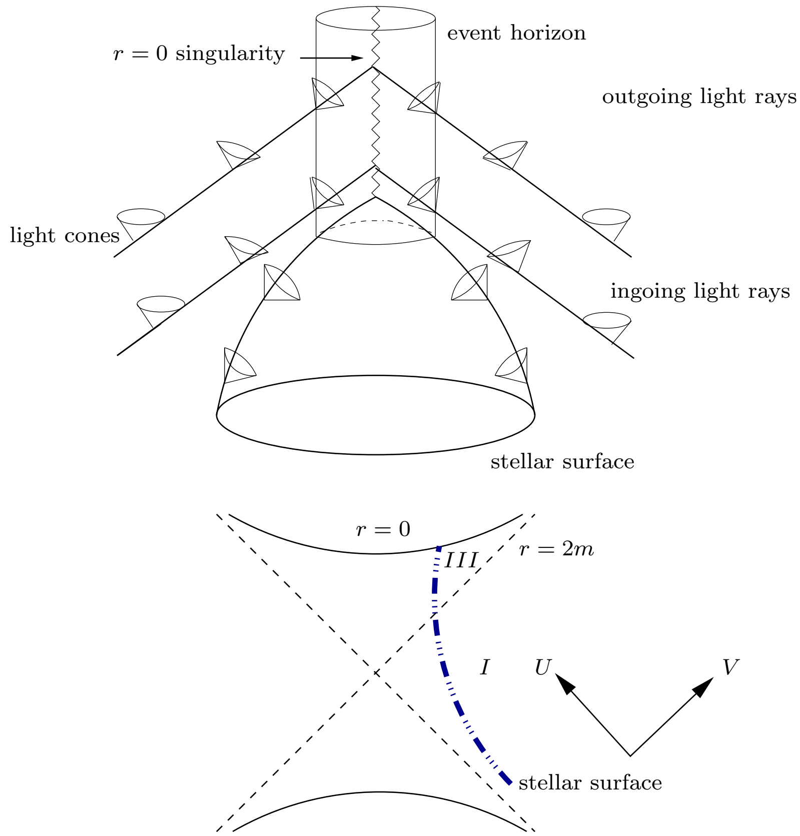

Figure 7 The spacetime of a collapsing star in (a) Eddington–Finkelstein coordinates and (b) Kruskal coordinates.

2.13 Embedding diagrams

The geometrical image of a cylinder as the surface of a tube is highly misleading in many respects. Intrinsically the cylinder has the geometry of flat space. The curvature one 'sees' when it is represented as a surface in three dimensional space comes from the embedding (the extrinsic geometry) of a 2-surface in three dimensional space. Nevertheless, the embedding picture of the cylinder does help to inform our feeling for what the geometry is like. To get some feel for the global geometry of the Schwarzschild black hole we can try to represent aspects of it by embeddings in three-space in a similar way.

To do this of course, we have to suppress all but two dimensions. Let us look at the spaces of constant time and also suppress one of the angular coordinates, by putting, say, $\theta = \pi/2$. The metric is

$$dl^2 = \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\phi^2.$$

The metric of the Euclidean embedding space is

$$dl^2 = dz^2 + dr^2 + r^2 d\phi^2,$$

which on $z = z(r)$ becomes

$$dl^2 = (1 + z'^2)dr^2 + r^2 d\phi^2,$$

where $z' = dz/dr$. These are the same if

$$z = 2(2m)^{1/2}(r - 2m)^{1/2}. \tag{2.56}$$

This gives the surface in Fig. 8.

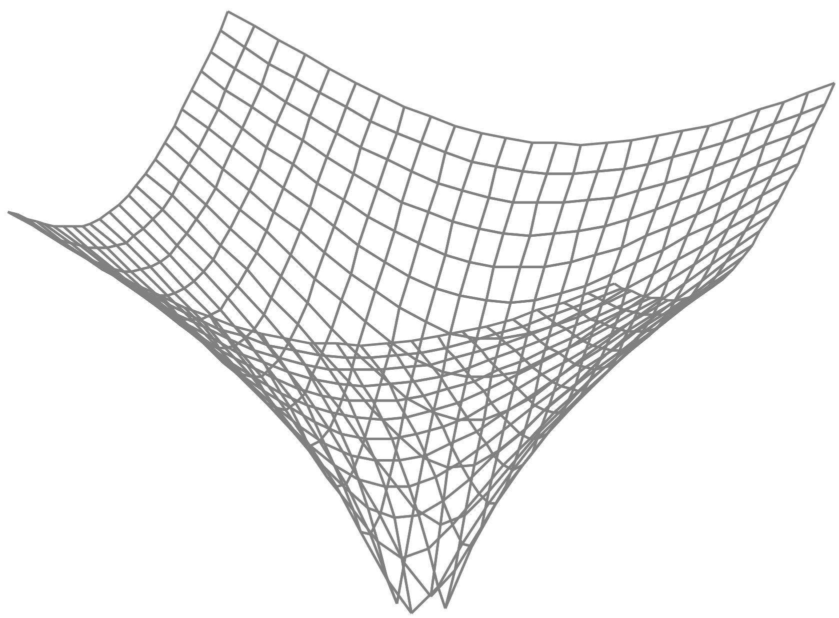

Figure 8 A plot of equation (2.56) showing the embedding surface with the same geometry as Schwarzschild in the $r - \phi$ plane.

This is the way in which black holes are customarily represented in popular literature. Since the Schwarzschild coordinates are not valid for $r = 2m$ we cannot follow the geometry further in by this means. If we look at the Kruskal picture however we see that through the bifurcation point at the origin the two Schwarzschild regions join together. For this cross-section (such as the one labelled $t =$ constant in Fig. 6 we get the Einstein-Rosen bridge as the embedding diagram, Fig. 9.

Although we cannot show it here, the only difference between this and a general section through the Kruskal metric, such as the one labelled $(U + V) =$ const. in Fig. 6, is that the throat region is extended and narrower, and breaks into two separate pieces if the section intersects $r = 0$.

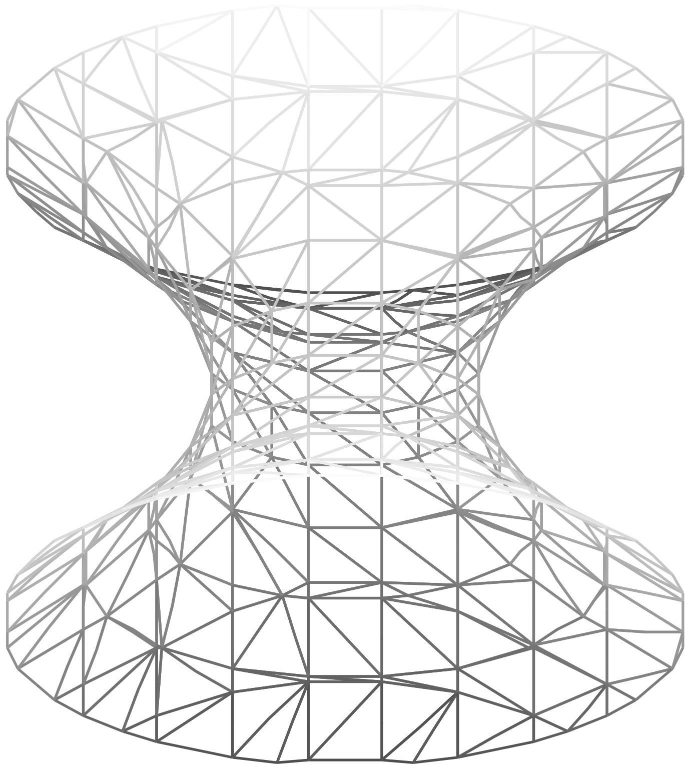

Figure 9 The Einstein–Rosen Bridge.

The two regions below and above the throat in Fig. 9 are two separate asymptotically flat universes. One might anticipate the possibility of travelling through the throat from one to the other. However, a glance at the full Kruskal picture shows that any attempt must end up in the singularity. In effect, the throat closes off faster than an observer can travel through it. On the other hand, it is not hard to imagine a picture in which the throat would connect two parts of the same Universe, in which case we have a *wormhole*. (See chapter 5.)

2.14 Asymptotic flatness

Far enough away from the central source of gravity the Schwarzschild spacetime approximates to Minkowski spacetime. Subject to certain technical definitions of what we mean by 'approximates to' such spacetimes are called asymptotically flat. For such spacetimes we can get what is sometimes a useful global picture of the causal structure (i.e. of which events can be causally connected) by using the Penrose–Carter diagrams. These make it possible to draw pictures of what is happening at infinity, by changing the coordinates to bring infinity to a finite coordinate value. This distortion is carried out in such a way that the relationship between light rays is maintained. We say that these diagrams show the causal structure of spacetimes (but clearly not their metric structure, since infinity is in the wrong place).

To see what this means consider the transformation from the Lorentzian coordinates (t, r, θ, ϕ) to new coordinates $(\psi, \xi, \theta, \phi)$ where

$$t = \frac{1}{2}\left(\tan\frac{1}{2}(\psi+\xi) + \tan\frac{1}{2}(\psi-\xi)\right) \tag{2.57}$$
$$r = \frac{1}{2}\left(\tan\frac{1}{2}(\psi+\xi) - \tan\frac{1}{2}(\psi-\xi)\right).$$

The Minkowski metric becomes

$$d\tau^2 = \Omega^{-2}[d\psi^2 - d\xi^2 - \sin^2\xi(d\theta^2 + \sin^2\theta d\phi^2)] = \Omega^{-2}d\tilde{\tau}^2,$$

where $\Omega = 2\cos\frac{1}{2}(\psi+\xi)\cos\frac{1}{2}(\psi-\xi)$. Neglecting the overall factor Ω we get a new metric $d\tilde{\tau}^2$ which has the same light paths as $d\tau^2$ (because the vanishing of one implies and is implied by the vanishing of the other). The replacement of the metric $d\tau^2$ by the metric $d\widetilde{\tau}^2$ is called a conformal transformation, with conformal factor Ω.

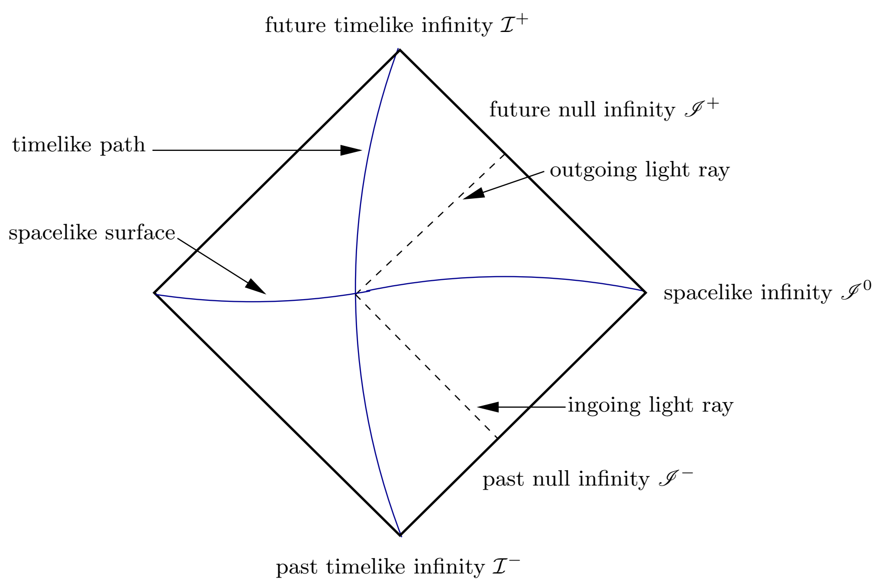

Figure 10 Asymptotic structure of Minkowski spacetime shown in a Penrose–Carter diagram.

The geometry of $d\tilde{\tau}^2$ is shown in Fig. 10. Light rays begin in the past at $\mathfrak{I}^-$ and end at $\mathfrak{I}^+$ (the lines $\psi + \xi = \pi$ and $\psi + \xi = -\pi$). These boundaries therefore represent light-like infinity. Causal relationships are respected by the transformations. Nevertheless, the paths of massive particles in the two metrics are not the same. All timelike paths in $d\widetilde{\tau}^2$ start from I^- and end at I^+. Arranging that a finite range $(-\pi, \pi)$ of the new coordinates ψ and ξ cover an infinite range of the original (t, r) coordinates in (2.57), and making the conformal transformation, enables us to form a finite picture of an infinite spacetime. Such pictures are called Penrose or Penrose–Carter diagrams. In the next section we consider the Penrose–Carter diagram for Schwarzschild spacetime.

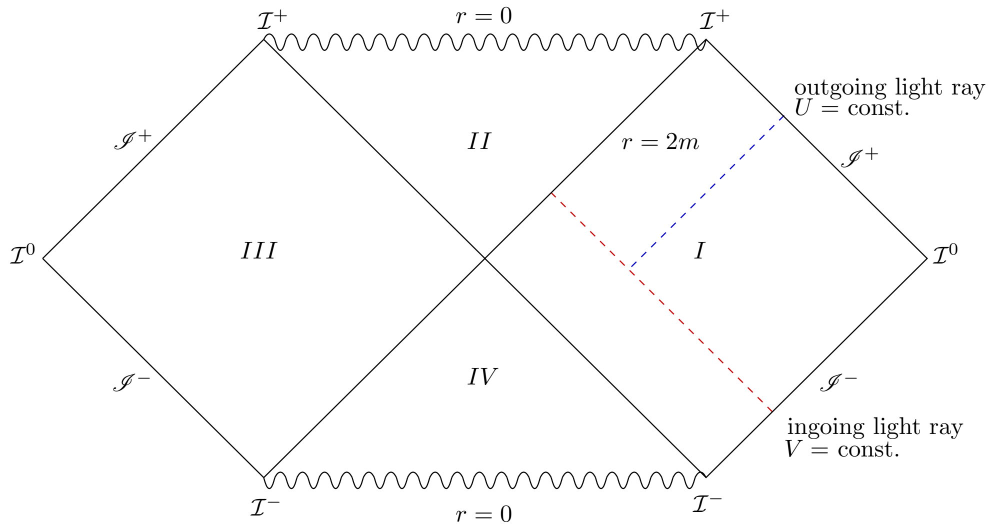

Figure 11 Penrose–Carter diagram of the Kruskal spacetime showing the asymptotic structure. Outgoing light rays end at null infinity on $\mathcal{I}^+$ and incoming rays begin at null infinity on $\mathcal{I}^-$. Spacelike infinity is denoted I^0 and timelike future and past infinity I^+ and $\bar{I}^-$ respectively. Light rays are given by $U =$ constant and $V =$ constant.

2.14.1 The Penrose–Carter diagram for the Schwarzschild metric

Our extension of the spacetime has taken us from the exterior region covered by the Schwarzschild coordinates to the 'whole' spacetime covered by the Kruskal coordinates. Although we have not proved it, in fact no further extensions are possible. We can put this together with a further conformal transformation that has no effect near the horizon, but which brings infinity to a finite distance. This provides us with a Penrose–Carter diagram of the black hole and allows us to picture its causal structure (Fig. 11).

The figure gives a view of the (U, V) plane.

2.14.2 The Penrose–Carter diagram for the Newtonian metric

Out of interest we can give a similar treatment for the Newtonian metric. The picture at infinity is the same as that for flat spacetime, but we need new coordinates to look at $r = 2m$. Confining ourselves to the (t, r) plane we put

$$dr = \pm \left(1 - \frac{2m}{r}\right)^{1/2} dR.$$

Then

$$d\tau^2 = \left(1 - \frac{2m}{r}\right)(dt^2 - dR^2).$$

The metric $d\widetilde{\tau}^2 = d\tau^2/(1 - 2m/r) = dt^2 - dR^2$ is therefore flat. Near $r = 2m$ we have

$$R \sim 2\sqrt{2m}(r - 2m)^{1/2},$$

so $r = 2m$ corresponds to $R = 0$. The metric is singular at $R = 0$, because $r < 2m$ clearly does not yield a Lorentzian metric. (Of course, near $r = 2m$ the Newtonian metric is not a physical gravitational field, because it is not even an approximate solution to Einstein's equations, but that does not stop us from considering the spacetime geometry to which is corresponds.)

2.15 Non-isolated black holes

There are several distinct concepts that coalesce in the Schwarzschild black hole, but which are different in general for black holes in a non-empty environment, for example for a black hole accreting material from the surrounding medium. In general we have to distinguish the following: the infinite redshift surface, the event horizon and the apparent horizon.

2.15.1 The infinite redshift surface

The surface defined by $g_{00} = 0$ has an infinite redshift to an external observer. To determine if a point belongs to the infinite redshift surface requires a knowledge of the metric just at that point, so this surface is a local construction. Thus it will not necessarily coincide with the event horizon in general.

2.15.2 Trapped surfaces

Consider a two dimensional spacelike surface and imagine that it emits a flash of light. A lightfront moves normal to the surface inwards and outwards. Under normal circumstances the area of the inward moving lightfront decreases and that of the outward moving one increases. Equivalently, the inward normals to the inward moving light front converge and the outward normals to the outward moving front diverge.

However, if we repeat this experiment inside a black hole, both sets of normals converge (Fig. 12). Our two dimensional spacelike surface is said to be trapped.

2.15.3 Apparent horizon

The outermost trapped surface is the apparent horizon. In the vacuum Schwarzschild solution this is the surface $r = 2m$. However, the apparent event horizon is clearly a local construction (you can measure the convergence or divergence locally), whereas the ('true') event horizon is a global property: it depends on constructing null geodesics to determine whether they reach infinity or not, which cannot be judged from the local behaviour.

For a concrete example consider a black hole accreting a spherical shell of matter (Frolov and Novikov, 1998). A light ray which was heading to infinity before the infall of the shell is subject to a stronger gravitational field after the matter is accreted, so may not in fact escape to infinity. Thus the original ray is part of the event horizon even though it was not initially trapped. The apparent horizon

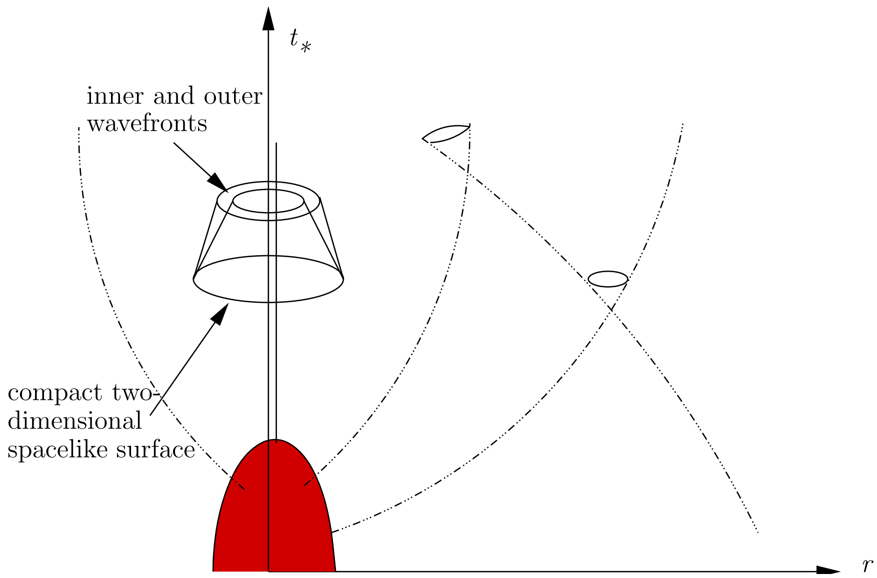

Figure 12 Both outgoing and ingoing wavefronts from a trapped surface converge. The trapped surface is inside a black hole formed by collapse.

therefore lies inside the event horizon. One can show that this is a general feature of the two horizons.

2.16 The membrane paradigm

Although the event horizon is not a physical boundary (and is locally undetectable) it is intuitively convenient to think of a black hole as a physical object, endowed with physical properties that encode its relation to the external world. To do this rigorously, we imagine the black hole surrounded by a membrane to which are attributed appropriate physical properties. This is the so-called 'membrane paradigm' (Thorne *et al.*, 1986). For example, to express the influence of a black hole on external electromagnetic fields the membrane is endowed with a surface resistivity allowing charges and currents to terminate electric and magnetic field lines. Similarly, to account for the deformations of a black hole in the presence of, for example, an orbiting planet, the membrane can be given certain elastic properties. Normal physical bodies have timelike surfaces. The membrane is therefore chosen not to be coincident with the horizon, which is a null surface, but as a timelike surface just outside the horizon. The black hole can then be thought of from the outside as a normal body in spacetime. This approach works not only for Schwarzschild black holes but also for the more general vacuum black holes of the next chapter, and also for black holes in general, which we do not consider here. The details are beyond the scope of this book but can be found in Thorne *et al.* (1986).