

## CHAPTER 1

# Weighted Residual Finite Element Method

There are a number of ways in which a finite element solution scheme for engineering problems can be constructed. Some of these techniques are problem specific and do not have universal applicability. For example, the displacement method commonly used by civil and mechanical engineers is more appropriate to cases where the problem requiring a solution can be stated in terms of a variational principle. Most types of process engineering problems, however, can only be formulated in terms of a set of governing differential equations derived from fundamental laws of physics. Therefore the development of finite element schemes in process engineering requires a more general approach. Weighted residual technique provides such a general approach for the development of efficient finite element schemes for the solution of the governing partial differential equations that represent many types of process engineering problems.

In this chapter, the basic concepts underpinning this approach are explained. We will mainly focus on the methodology and procedures used for the development of numerical solution schemes and will avoid discussions related to the fundamental mathematical theory of the method.

### 1.1 Basic Concept

Numerical solution of differential equations arising in engineering problems is usually based on finite difference, finite element, boundary element or finite volume techniques. Other numerical methods may also be used to solve specific problems. In general, the finite element method has a greater geometrical flexibility than other currently available numerical methods. It can also cope very effectively with a wide range of boundary conditions (Nassehi, 2002). The general weighted residual method is the basic technique used to construct finite element schemes for field problems. Therefore we start with a brief description of this method.

Consider a boundary value problem represented as

$$\begin{cases} LT = f & \text{in } \Omega \\ T = a & \text{on } \Gamma \end{cases} \quad (1.1)$$

where  $\Omega$  is the problem domain and  $\Gamma$  is its boundary, in the absence of an analytical solution an approximate representation of  $T$  can be written as

$$T \approx \tilde{T} = a + \sum_{i=1}^m \alpha_i \varphi_i \quad (1.2)$$

where  $\alpha_i$  is a set of constant coefficients and  $\varphi_i$  represents a set of geometrical functions called basis functions. Substituting Eq. (1.2) into Eq. (1.1) we have

$$L\tilde{T} - f = R_\Omega \quad (1.3)$$

where  $R_\Omega \neq 0$  is the residual which will inevitably appear through the insertion of an approximation instead of an exact solution for the field variable into the differential equation. This equation can now be written as

$$L \left[ a + \sum_{i=1}^m \alpha_i \varphi_i \right] - f = R_\Omega \quad (1.4)$$

The weighted residual method is based on the elimination of this residual. To achieve this, the residual is weighted by appropriate position-dependent functions and integrated over the domain to obtain a statement as

$$\int_{\Omega} W_j R_\Omega d\Omega = 0 \quad j = 1, 2, 3, \dots, m \quad (1.5)$$

where  $W_j$  are linearly independent weight functions. The above equation can be rewritten as

$$\int_{\Omega} W_j \left\{ L \left[ a + \sum_{i=1}^m \alpha_i \varphi_i \right] - f \right\} d\Omega = 0 \quad j = 1, 2, 3, \dots, m \quad (1.6)$$

This equation represents the weighted residual statement of the original differential equation.

### 1.1.1 Practical procedure

As we see from the outlined concept, the weighted residual solution of the described problem depends on the use of appropriate weight and basis functions. In order to develop practical solution schemes for field problems these functions need to be determined in a systematic and rigorous manner.

The problem of finding weight functions is more straightforward and we explain it first. To simplify our explanations of the technique, here we use an example to outline the steps required to obtain a weighted residual statement for a typical partial differential equation.

Consider the following transient convection–diffusion equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f - \nabla \cdot k \nabla f - \dot{q} = 0 \tag{1.7}$$

where  $f$  is the unknown function (i.e. the field variable),  $t$  is time,  $\mathbf{v}$  is the velocity vector,  $k$  is a diffusion coefficient, and  $\dot{q}$  is a source/sink term. We can start by approximating the time dependent term in Eq. (1.7) using a simple finite difference relationship such as a forward difference. Therefore

$$\frac{f^{n+1} - f^n}{\Delta t} + \mathbf{v} \cdot \nabla f^{n+\theta} - \nabla \cdot k \nabla f^{n+\theta} - \dot{q} = 0 \tag{1.8}$$

This means that the field variable in Eq. (1.8), given at a time level  $n + \theta$ ,  $0 \leq \theta \leq 1$ , is discretized using a forward difference formula. If we now substitute the function  $f$  in Eq. (1.7) with an approximate form analogous to Eq. (1.2) an error is generated. Therefore representing  $\tilde{f}$  in terms of basis functions (i.e.  $\tilde{f} = \sum_{i=1}^m \varphi_i f_i$ ) the following least squares functional can be constructed

$$I_{n+1} = \int_{\Omega} \left[ \frac{\sum_{i=1}^m \varphi_i f_i^{n+1} - \sum_{i=1}^m \varphi_i f_i^n}{\Delta t_n} + \mathbf{v} \cdot \nabla \left( \sum_{i=1}^m \varphi_i f_i^{n+\theta} \right) - \nabla \cdot k \nabla \left( \sum_{i=1}^m \varphi_i f_i^{n+\theta} \right) - \dot{q} \right]^2 d\Omega = \int_{\Omega} [f_{\text{res}}^l]^2 d\Omega \tag{1.9}$$

where  $n$  is the time level. Minimization of functional (1.9) with respect to  $f_i$  gives

$$\frac{\partial I}{\partial f_i} = 2 \int_{\Omega_e} \left( \frac{\varphi_i}{\Delta t_n} + \theta \mathbf{v} \cdot \nabla \varphi_i - \theta \nabla \cdot k \nabla \varphi_i \right) [f_{\text{res}}^l] d\Omega = 0 \tag{1.10}$$

and

$$\int_{\Omega} (\varphi_i + \theta \Delta t_n \mathbf{v} \cdot \nabla \varphi_i - \theta \Delta t_n \nabla \cdot k \nabla \varphi_j) [f_{\text{res}}^l] d\Omega = 0 \tag{1.11}$$

or

$$\int_{\Omega} W [f_{\text{res}}^l] d\Omega = 0 \tag{1.12}$$

which is similar to Eq. (1.5). We see that Eq. (1.11) represents a weighted residual statement where the weighting function is given as

$$W = \varphi_i + \theta \Delta t_n \mathbf{v} \cdot \nabla \varphi_i - \theta \Delta t_n \nabla \cdot k \nabla \varphi_i \quad (1.13)$$

By selecting various terms of the weight function given in Eq. (1.13) the following weighted residual methods can be developed

- The standard (Babnov) Galerkin method is obtained if we only keep the first term in Eq. (1.13). It can hence be readily ascertained that in this method the basis and weight functions are identical.
- A first order Petrov–Galerkin scheme is obtained by neglecting the third term in Eq. (1.13). This method corresponds to a technique called Stream Line Upwind Petrov–Galerkin (SUPG) scheme, a special form of this scheme (called Inconsistent Upwinding) in which the second term in the weight function is only applied to the weighting of the convection term is also used (however, note that the inconsistent upwinding cannot be regarded as a true weighted residual method).
- If we retain all of the terms in the weight function, a scheme corresponding to a second order Petrov–Galerkin formulation is obtained.

In steady state problems the time-dependent term of the residual is eliminated. Therefore, for steady state problems a scheme equivalent to the combination of Galerkin and least square methods is obtained.

The procedure leading to the development of practical weighted residual schemes does not provide a systematic technique for the derivation or selection of appropriate basis functions. This has been a fundamental difficulty with the use and application of these theoretically powerful techniques in practical problems. However, combination of these concepts with the finite element discretization resolves all such problems in a very efficient manner. In the following sections we describe the procedures used to develop finite element approximations and give examples of the solution of differential equations by the weighted residuals finite element schemes.

### 1.1.2 Finite element approximations

The first step in the formulation of a finite element approximation for a field problem is to divide the problem domain into a number of smaller sub-regions without leaving any gaps or allowing any overlapping between them. This process is called “Domain Discretization.” An individual sub-region in a discretized domain is called a “finite element” and collectively, the finite elements provide a “finite element mesh” for the discretized domain.

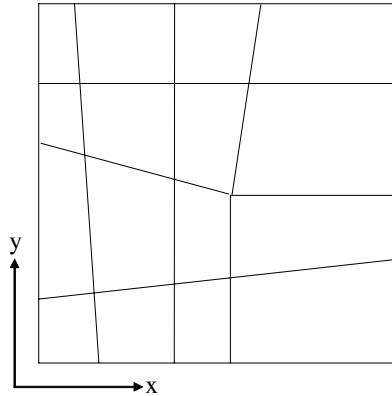


Fig. 1.1 A finite element mesh of four node quadrilateral elements.

In general, the elements in a finite element mesh may have different sizes but all of them usually have a common basic shape (e.g. they are all lines, triangles or quadrilaterals, prisms or tetrahedrons in one-, two-, and three-dimensional domains, respectively) and have an equal number of nodes (Fig. 1.1).

The nodes are the sampling points in an element where the numerical values of the unknowns will be calculated. All types of finite elements should have some nodes located on their boundary lines. Some of the commonly used finite elements also have interior nodes. Boundary nodes of the individual finite elements appear as the junction points between the elements in a finite element mesh.

In most engineering problems the boundary of the problem domain includes curved sections. The discretization of domains with curved boundaries using meshes that consist of elements with straight sides inevitably involves some error. This type of discretization error can obviously be reduced by mesh refinements. Mesh refinement simply means that the size of the elements that make up the mesh is reduced and a curved boundary can be approximated more accurately. However, in general, such a discretization error cannot be entirely eliminated unless finite elements which themselves have curved sides are used.

After the discretization of a problem domain into finite elements, an unknown field function, anywhere inside an element, can be approximately calculated as an interpolant of its values at the nodes of that element. Nodal values of a field unknown (called the nodal degrees of freedom) therefore appear as coefficients multiplied to geometrical interpolation

(shape) functions. Obviously the form of geometrical interpolation functions associated with a finite element will depend on its shape and the number of nodes that can be selected as the sampling for that element. Different types of finite elements have therefore been developed over the past decades to generate desired approximations for a wide range of problems.

The described interpolation provides a systematic way of deriving approximate forms within a finite element for field unknowns in terms of their nodal values and set of geometrical functions. As these approximations are derived on an elemental space their use to form weighted residual statements results on an elemental statement. After this initial step, however, weighted residual statements for all of the elements in a computational grid can be formed and assembled together to obtain a global system of equations whose solution yields the values of nodal degrees of freedom.

Further explanations regarding the procedure for the derivation of interpolation functions, inter-element continuity of variables and elemental residual statements are explained in Chap. 2.

## 1.2 Numerical Integration

As demonstrated in Eq. (1.6), weighted residual solution of differential equations requires function integrations. However, after the domain discretization into finite elements the required integrations are confined to the space of individual elements. Therefore the functions that need to be integrated originate from geometrical interpolation functions. This means that the integrands are often low degree polynomials. Therefore they can be accurately evaluated using simple numerical integration (quadrature) techniques. The main advantage of using quadrature is that these techniques can be incorporated into a computer program rendering the entire finite element solution procedure automatic.

Consider the integration of a function  $f(x_1, x_2)$  over a quadrilateral element within a finite element mesh, as shown in Fig. 1.2, we have

$$I = \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} f(x_1, x_2) dx_1 dx_2 \quad (1.14)$$

where  $x_1$  and  $x_2$  are the global coordinates and  $\alpha_1$ ,  $\alpha_2$ ,  $\beta_1$ , and  $\beta_2$  are the limits of integration.

Integration limits in Eq. (1.14) will change from element to element in a mesh if global coordinates are used. This makes the finite element procedure cumbersome. However, as shown in Fig. 1.2, mapping of elements from the global mesh onto a master element of regular shape (in this case

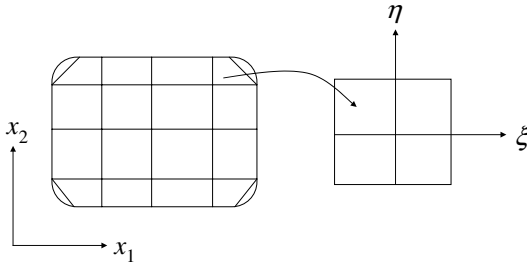


Fig. 1.2 Transformation from a global system to a master element with locally defined coordinates.

a square because the elements in the original mesh are two-dimensional quadrilaterals and can be mapped onto a square) resolves this difficulty. After such a transformation, the required integrals can first be calculated using uniform limits over the master element and then transformed back to the original system. Further simplifications can also be considered. For example, the use of a normalized local coordinate system corresponds to the uniform integration limits of  $-1$  to  $+1$ . This allows the utilization of a simple numerical integration method called the Gauss–Legendre formula (Gerald and Wheatley, 1984).

An additional advantage of the described mapping is to overcome the problem of curved boundaries. Using elements which have straight sides inevitably results in discretization errors (see the gap between the corner elements in Fig. 1.2 and the actual domain boundary). However, elements with curved sides can also be mapped into a master element of regular shape. A fuller explanation of this point is outlined below.

### 1.2.1 Mapping of irregular and curved elements onto master elements

A one-to-one transformation (mapping) between any global and local coordinate systems, respectively representing actual finite element mesh and a selected master element, can be established using a variety of techniques. The most general method is a form of “parametric mapping” in which the transformation functions are the polynomials based on the element shape functions (Nassehi, 2002).

A graphical example of mapping between an irregular shape element and a regular master element is shown in Fig. 1.3. The transformation is

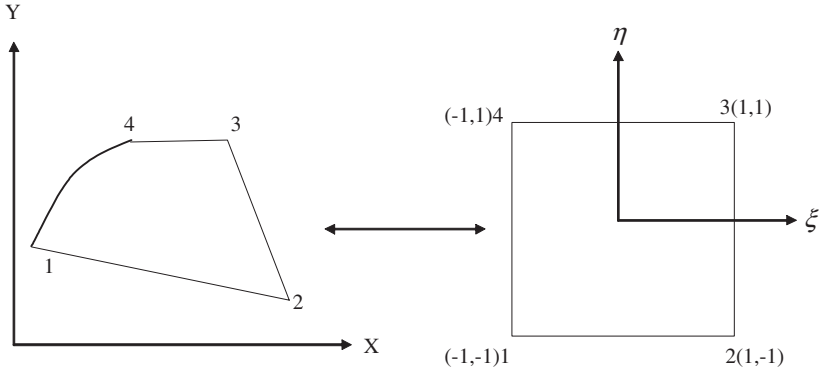


Fig. 1.3 Isoparametric mapping of an irregular element.

represented as

$$\mathbf{M} = \begin{cases} x = x(\xi, \eta) \\ y = y(\xi, \eta) \end{cases} \quad (1.15)$$

In “Isoparametric” mapping, which is the most commonly used form of parametric mapping, the coordinate transformation functions are identical to the interpolation functions associated with the selected finite elements. However, it should be noted that the linear shape functions in  $\xi$  and  $\eta$  can only be used to map irregular elements with straight sides to a master element.

In general, elements with curved sides can only be generated using higher-order master elements. Isoparametric transformation functions between a global coordinate system and local coordinates are, in general, written as

$$\begin{cases} x = \sum_{i=1}^p N_i(\xi, \eta) x_i \\ y = \sum_{i=1}^p N_i(\xi, \eta) y_i \end{cases} \quad (1.16)$$

where  $x_i$  and  $y_i$  are the nodal coordinates in the global system. Implementation of isoparametric mapping is based on the expression of the derivatives of shape functions in terms of local variables. Derivatives of a function  $T$  in terms of local variables  $(\xi, \eta)$  can be expressed in terms of its global derivatives, given with respect to global  $(x, y)$  coordinates, using the rules

of differentiation as

$$\begin{cases} \frac{\partial T}{\partial \xi} = \frac{\partial T}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial T}{\partial y} \frac{\partial y}{\partial \xi} \\ \frac{\partial T}{\partial \eta} = \frac{\partial T}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial T}{\partial y} \frac{\partial y}{\partial \eta} \end{cases} \quad (1.17)$$

Therefore using matrix notation

$$\begin{bmatrix} \frac{\partial T}{\partial \xi} \\ \frac{\partial T}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \end{bmatrix} \quad (1.18)$$

or

$$\begin{bmatrix} \frac{\partial T}{\partial \xi} \\ \frac{\partial T}{\partial \eta} \end{bmatrix} = J \begin{bmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \end{bmatrix} \quad (1.19)$$

where matrix  $J$  is called the ‘‘Jacobian’’ of coordinate transformation. Therefore

$$\begin{bmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial T}{\partial \xi} \\ \frac{\partial T}{\partial \eta} \end{bmatrix} \quad (1.20)$$

we can hence write

$$\begin{bmatrix} \frac{\partial \tilde{T}}{\partial x} \\ \frac{\partial \tilde{T}}{\partial y} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial \sum_{i=1}^p N_i(\xi, \eta) T_i}{\partial \xi} \\ \frac{\partial \sum_{i=1}^p N_i(\xi, \eta) T_i}{\partial \eta} \end{bmatrix} \quad (1.21)$$

Obviously the existence of an inverse for the Jacobian matrix is a necessary requirement for the described transformation. It should also be noted that the mapping of irregular elements into master elements generates some degree of approximation in finite element calculations. In general, mapping of highly distorted elements generates a high degree of mapping error and in extreme cases the sign of the Jacobian may change during transformation, rendering the operation invalid. In certain types of finite element approximations, transformation of second-order derivatives is necessary. Isoparametric transformation of second- or higher-order derivatives is not trivial and needs lengthy manipulations. Details of a procedure for transformation of second-order derivatives are represented in Petera *et al.* (1993).

Transformation of Eq. (1.14) into the square element shown in Fig. 1.3 gives

$$I = \int_{-1}^1 \int_{-1}^1 F(\xi, \eta) \det \mathbf{J}^e d\xi d\eta \quad (1.22)$$

where  $\mathbf{J}^e$  is the Jacobian of coordinate transformation (Spiegel, 1974) and the limits of integration are defined by the local coordinates. Therefore

$$dx_1 dx_2 = \det \mathbf{J}^e d\xi d\eta \quad (1.23)$$

After some algebraic manipulations we can write

$$I = \int_{-1}^1 \int_{-1}^1 G(\xi, \eta) d\xi d\eta \quad (1.24)$$

According to the Gauss–Legendre quadrature  $I$  is calculated as

$$I = \int_{-1}^1 \int_{-1}^1 G(\xi, \eta) d\xi d\eta \approx \sum_{i=1}^m \sum_{j=1}^n G(\xi_i, \eta_j) w_i w_j \quad (1.25)$$

where  $\xi_i$  and  $\eta_j$  are the quadrature point coordinates,  $w_i$  and  $w_j$  are the corresponding weight factors and  $m$  and  $n$  are the number of quadrature points in each summation.

The number of quadrature points in these summations depends on the order of the polynomial function in the integral. In one-dimensional problems this quadrature yields an exact result for a polynomial of degree  $2n - 1$  (or less) using  $n$  points. Integrands in elemental equations are based on interpolation functions, which are low-order polynomials. Therefore the number of required quadrature points in practical calculations is low (usually  $n = 2$  or  $3$ ).

The described example can be readily implemented in quadrilateral elements and its generalization to three-dimensional brick type elements with 8, 20 or 27 nodes is straightforward. However, there are many applications in which the domain discretization into quadrilateral elements or their three-dimensional counterparts creates difficulties. In these cases triangular and tetrahedral elements for two- and three-dimensional problems are used. It is not possible to choose an element-based coordinate system for triangular elements in which limits of integration remain between  $-1$  and  $+1$ . Therefore, the described Gauss–Legendre quadrature is not a suitable method for such elements. However, this problem can be resolved using properly defined “area coordinates” in appropriately selected triangular master elements. Detailed definition of area coordinates and the technique for developing quadrature techniques for triangular elements are given by

Zienkiewicz and Taylor (1994) who list the coordinates of the sampling points and their corresponding weights. The coordinates and integration limits in these elements are therefore defined with respect to area coordinates. After the definition of the local coordinate system in terms of area coordinates a straightforward quadrature procedure for these elements can be constructed. Tables 1.1–1.3 show the sampling points and weighting factors, respectively, for quadrilateral, triangular, and tetrahedral elements.

Using the data provided in Table 1.2, a polynomial function of order  $q(\xi^i \eta^j$  with  $i + j \leq q$ ) can be calculated as

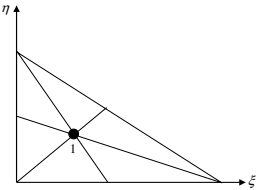
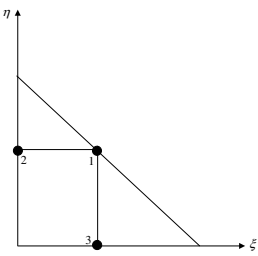
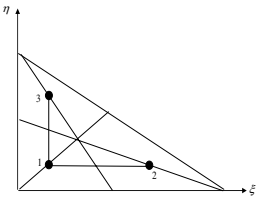
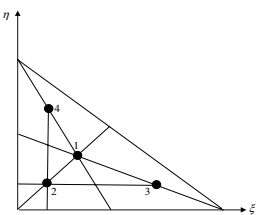
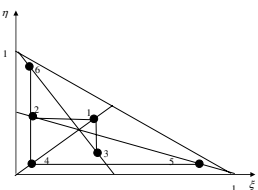
$$\int_0^1 \int_0^{1-\xi} G(\xi, \eta) d\eta d\xi \cong \sum_{i=1}^n w_i G(\xi_i, \eta_i)$$

in which  $n$  is the number of quadrature points (Dhatt and Touzot, 1985).

Table 1.1. Quadrature points and weights for quadrilateral elements.

$m = n$	$\xi_i, \eta_i$	$w_i$
2	$\pm 0.57735027$	1
3	0	0.88888889
	$\pm 0.77459667$	0.55555555
4	$\pm 0.33998104$	0.65214515
	$\pm 0.86113631$	0.34785485
5	0	0.56888889
	$\pm 0.53846931$	0.47862867
	$\pm 0.90617985$	0.23692689
6	$\pm 0.23861918$	0.46791393
	$\pm 0.66120939$	0.36076157
	$\pm 0.93246951$	0.17132449
7	0	0.41795918
	$\pm 0.40584515$	0.38183005
	$\pm 0.74153119$	0.27970539
	$\pm 0.94910791$	0.12948497
8	$\pm 0.18343464$	0.36268378
	$\pm 0.52553241$	0.31370665
	$\pm 0.79666648$	0.22238103
	$\pm 0.96028986$	0.10122854
10	$\pm 0.14887434$	0.29552422
	$\pm 0.43339539$	0.26926672
	$\pm 0.67940957$	0.21908636
	$\pm 0.86506337$	0.14945135
	$\pm 0.97390653$	0.06667134

Table 1.2. Quadrature points and weights for triangular elements.

	Order $m$	Number of points	Coordinates		Weights, $w_i$
			$\xi_i$	$\eta_i$	
	1	1	1/3	1/3	1/2
	2	3	1/2 0 1/2	1/2 1/2 0	1/6
	2	3	1/6 2/3 1/6	1/6 1/6 2/3	1/6
	3	4	1/3 1/5 3/5 1/5	1/3 1/5 1/5 3/5	-27/96 25/96
	4	6	$a$ $1-2a$ $a$ $b$ $1-2b$ $b$	$a$ $a$ $1-2a$ $b$ $b$ $1-2b$	0.11169079 0.05497587
			$a = 0.445948$		
			$b = 0.091576$		

(Continued)

Table 1.2. (Continued)

	Order $m$	Number of points	Coordinates		Weights, $w_i$
			$\xi_i$	$\eta_i$	
	5	7	$1/3$	$1/3$	$9/80$
	$a = \frac{6 + \sqrt{15}}{21}$		$a$	$a$	
	$= 0.470142$		$1-2a$	$a$	$0.06619707$
			$a$	$1-2a$	
	$b = \frac{4}{7} - a$		$b$	$b$	
	$= 0.101285$		$1-2b$	$b$	$0.06296959$
			$b$	$1-2b$	
			$a$	$a$	
			$1-2a$	$a$	$0.02542245$
			$a$	$1-2a$	
			$b$	$b$	
			$1-2b$	$b$	$0.05839314$
		$b$	$1-2b$		
		$c = 0.310352$	$c$	$d$	
		$d = 0.053145$	$d$	$c$	
			$1 - (c+d)$	$c$	$0.04142554$
			$1 - (c+d)$	$d$	
			$c$	$1 - (c+d)$	
			$d$	$1 - (c+d)$	

The corresponding quadrature formula for a tetrahedral element is given as

$$\int_0^1 \int_0^{1-\xi} \int_0^{1-\xi-\eta} G(\xi, \eta, \zeta) d\zeta d\eta d\xi \cong \sum_{i=1}^n w_i G(\xi_i, \eta_i, \zeta_i)$$

In which  $n$  is the number of quadrature points (Dhatt and Touzot, 1985).

### 1.3 Steps Used to Obtain a Finite Element Solution for a Field Problem

Based on the explanations we have given up to this point it is now possible to outline the steps used to develop a weighted residual finite element solution for a field problem.

Table 1.3. Quadrature points and weights for tetrahedral elements.

Order $m$	Number of points $r$	Coordinates			Weights $w_i$
		$\xi_i$	$\eta_i$	$\zeta_i$	
1	1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$
	4				
2	$a = \frac{5 - \sqrt{5}}{20}$	$a$	$a$	$a$	$\frac{1}{24}$
	$b = \frac{5 + 3\sqrt{5}}{20}$	$a$	$b$	$b$	
	5				
3	$a = \frac{1}{4}$	$a$	$a$	$a$	$-\frac{2}{15}$
	$b = \frac{1}{6}$	$b$	$b$	$b$	
	$c = \frac{1}{2}$	$b$	$c$	$b$	
	15				
5	$a = \frac{1}{4}$	$a$	$a$	$a$	$\frac{8}{405}$ or $\frac{112}{5670}$
	$a = \frac{1}{4}$	$b_i$	$b_i$	$b_i$	
	$b_1$	$b_i$	$b_i$	$c_i$	
	$b_2 = \frac{7 - \sqrt{15}}{34}$	$b_i$	$c_i$	$b_i$	
	$c_1 = \frac{13 + 3\sqrt{15}}{34}$	$c_i$	$b_i$	$b_i$	
	$d = \frac{5 - \sqrt{15}}{20}$	$d$	$d$	$e$	
	$e = \frac{5 + \sqrt{15}}{20}$	$d$	$e$	$d$	
		$e$	$d$	$d$	
		$e$	$e$	$e$	
		$e$	$e$	$d$	
				$\frac{w_1}{w_2} = \frac{2665 + 14\sqrt{15}}{226800}$	

- Step 1. The global solution domain is discretized into a number of suitable finite elements.
- Step 2. The field variables in the governing equations are substituted by approximations in terms of the interpolation functions associated with the type of finite elements used to discretize the domain.
- Step 3. An elemental weighted residual statement is constructed.
- Step 4. If necessary, by the application of procedures such as integration by parts (or in general Green's theorem), the order of the

differentials in the formed weighted residual equations are reduced to make the approximations compatible with the elements used.

- Step 5. Using matrix notations, elemental stiffness equations representing the described weighted residual forms are derived.
- Step 6. Elemental stiffness equations are assembled into a global system of algebraic equations over their common nodes. In this step isoparametric mapping can be used to transform the derived equations into a local natural coordinate system. Note that this will make it possible to use quadrature to evaluate integrals in the elemental equations. Therefore, the procedure will be commonly employed irrespective of the shape of the elements in the global mesh.
- Step 7. Boundary conditions are incorporated into the assembled global system of equations to make it determinate.
- Step 8. The global set of equations is solved using an appropriate solution technique for the set of algebraic equations.

In the following sections we present a number of solved examples in which step-by-step application of the outlined procedure is described.

### 1.3.1 Example 1

We start with the standard Galerkin finite element solution of the following one-dimensional steady state problem. Consider a one-dimensional domain as shown in Fig. 1.4.

The following differential equation represents a molecular process (e.g. heat conduction) along this domain and its finite element solution should provide discrete numerical results for the field unknown ( $T$ ) on a number of selected internal points.

$$\frac{d^2 T}{dx^2} + T = 10 \quad \text{in } \Omega \quad \text{Subject to} \quad (1.26)$$

$$T_A(x = 0) = 1 \quad \text{and} \quad T_B(x = 1) = 4$$



Fig. 1.4 Domain  $\Omega$  defined by line AB.

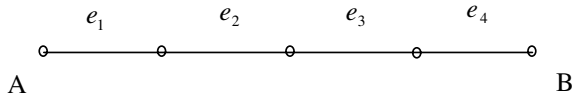


Fig. 1.5 Discretization of  $\Omega$  into finite elements.

To formulate a standard Galerkin finite element solution for Eq. (1.26) we follow the steps described in the previous section.

*Step 1. Discretization of the problem domain*

The domain  $\Omega$  is discretized into a mesh of four equal size finite elements, as shown in Fig. 1.5.

*Step 2. Approximation using shape functions*

Within the space of each finite element, the unknown function is approximated using interpolation functions corresponding to the two-node (linear) Lagrange elements as

$$\tilde{T} = \sum_{i=1}^2 N_i(x)T_i \quad (1.27)$$

Where  $N_i(x)$ ,  $i = 1, 2$  are the shape functions and  $T_i$ ,  $i = 1, 2$  are the nodal degrees of freedom (i.e. nodal unknowns).

*Step 3. Formulation of elemental weighted residual statement*

The residual obtained via the insertion of  $\tilde{T}$  into the differential equation is weighted and integrated over each element as

$$\int_{\Omega_e} w \left( \frac{d^2 \tilde{T}}{dx^2} + \tilde{T} - 10 \right) dx = 0 \quad (1.28)$$

where  $w$  is a weighting function. In the standard Galerkin method the selected weight functions are identical to the shape functions and hence Eq. (1.28) is written as

$$\int_{\Omega_e} w_j \left[ \frac{d^2 \sum_{i=1}^2 N_i(x)T_i}{dx^2} + \sum_{i=1}^2 N_i(x)T_i - 10 \right] dx = 0 \quad (1.29)$$

*Step 4. Integration by parts (Green's theorem)*

At this stage the formulated Galerkin-weighted residual equation (1.29) contains second-order derivatives. Therefore, using elements whose associated interpolation functions are only first order cannot generate an acceptable solution for this equation. To solve this difficulty the second-order derivative term in Eq. (1.29) is integrated by parts to obtain the “weak” form of the weighted residual statement as

$$\left. \frac{w_j d \sum_{i=1}^2 N_i(x) T_i}{dx} \right|_{\Gamma_e} - \int_{\Omega_e} \left( \frac{d \sum_{i=1}^2 N_i(x) T_i}{dx} \cdot \frac{dw_j}{dx} \right) dx + \int_{\Omega_e} w_j \sum_{i=1}^2 N_i(x) T_i dx - 10 \int_{\Omega_e} w_j dx = 0 \tag{1.30}$$

where  $\Gamma_e$  represents an element boundary.

*Step 5. Formulation of the elemental stiffness equations*

The weight function used in the Galerkin formulation can be identical to either of the shape functions of a two-node linear element, therefore, for each weight function an equation corresponding to the weak statement (1.30) is derived.

$$\left\{ \begin{array}{l} w_1 \frac{d(N_1 T_1 + N_2 T_2)}{dx} \Big|_{\Gamma_e} - \int_{\Omega_e} \frac{dw_1}{dx} \frac{d(N_1 T_1 + N_2 T_2)}{dx} dx \\ \quad + \int_{\Omega_e} w_1 (N_1 T_1 + N_2 T_2) dx - \int_{\Omega_e} 10 w_1 dx = 0 \\ w_2 \frac{d(N_1 T_1 + N_2 T_2)}{dx} \Big|_{\Gamma_e} - \int_{\Omega_e} \frac{dw_2}{dx} \frac{d(N_1 T_1 + N_2 T_2)}{dx} dx \\ \quad + \int_{\Omega_e} w_2 (N_1 T_1 + N_2 T_2) dx - \int_{\Omega_e} 10 w_2 dx = 0 \end{array} \right. \tag{1.31}$$

Using matrix notation, Eq. (1.31) is written as

$$\left[ \begin{array}{cc} - \int_{\Omega_e} \left( \frac{dw_1}{dx} \frac{dN_1}{dx} - w_1 N_1 \right) dx & - \int_{\Omega_e} \left( \frac{dw_1}{dx} \frac{dN_2}{dx} - w_1 N_2 \right) dx \\ - \int_{\Omega_e} \left( \frac{dw_2}{dx} \frac{dN_1}{dx} - w_2 N_1 \right) dx & - \int_{\Omega_e} \left( \frac{dw_2}{dx} \frac{dN_2}{dx} - w_2 N_2 \right) dx \end{array} \right] \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} -w_1 \Phi|_{\Gamma_e} + \int_{\Omega_e} 10 w_1 dx \\ -w_2 \Phi|_{\Gamma_e} + \int_{\Omega_e} 10 w_2 dx \end{Bmatrix} \tag{1.32}$$

In which  $\Phi$  represents the boundary line term. Although the elemental stiffness equation (1.32) has a common form for all of the elements in the mesh, its utilization based on the shape functions defined in the global coordinate system is not convenient. The problem can be clearly demonstrated considering that the use of a global coordinate system results in having different interpolation functions for different elements. For example, approximate value of the field unknown anywhere inside element 1 is determined in terms of its nodal values by the following Lagrangian interpolation

$$\tilde{T}(x) = \frac{0.25 - x}{0.25}T_1 + \frac{x}{0.25}T_2$$

Therefore interpolation functions associated with element  $e_1$  are

$$N_1 = \frac{0.25 - x}{0.25} \quad \text{and} \quad N_2 = \frac{x}{0.25}$$

However, following a similar procedure the field unknown within element 2 is interpolated as

$$\tilde{T}(x) = \frac{0.5 - x}{0.25}T_2 + \frac{x - 0.25}{0.25}T_3$$

Consequently interpolation functions corresponding to element  $e_2$  in this example are given as

$$N_1 = \frac{0.5 - x}{0.25}, \quad N_2 = \frac{x - 0.25}{0.25}$$

and so on. Furthermore, in a global system, limits of definite integrals in the coefficient matrix will be different for each element. The described difficulties are readily resolved using a local coordinate system to define the elemental interpolation functions for a two node linear element as

$$\begin{cases} N_1 = \frac{\ell - x}{\ell} \\ N_2 = \frac{x}{\ell} \end{cases} \quad \text{which give} \quad \begin{cases} \frac{dN_1}{dx} = \frac{-1}{\ell} \\ \frac{dN_2}{dx} = \frac{1}{\ell} \end{cases}$$

where  $\ell$  is the element length. Therefore, Eq. (1.32) is written as

$$\begin{aligned} & \left[ -\int_0^\ell \left( \frac{dw_1}{dx} \frac{dN_1}{dx} - w_1 N_1 \right) dx - \int_0^\ell \left( \frac{dw_1}{dx} \frac{dN_2}{dx} - w_1 N_2 \right) dx \right] \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} \\ & \left[ -\int_0^\ell \left( \frac{dw_2}{dx} \frac{dN_1}{dx} - w_2 N_1 \right) dx - \int_0^\ell \left( \frac{dw_2}{dx} \frac{dN_2}{dx} - w_2 N_2 \right) dx \right] \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} \\ & = \begin{Bmatrix} -w_1 \Phi|_{\Gamma_e} + \int_0^\ell 10w_1 dx \\ -w_2 \Phi|_{\Gamma_e} + \int_0^\ell 10w_2 dx \end{Bmatrix} \end{aligned} \quad (1.33)$$

Substitution of locally defined interpolation functions into Eq. (1.33) gives

$$\frac{1}{(\ell)^2} \begin{bmatrix} \int_0^\ell (-1 + \ell^2 - 2\ell x + x^2) dx & \int_0^\ell (1 + \ell x - x^2) dx \\ \int_0^\ell (1 + \ell x - x^2) dx & \int_0^\ell (-1 + x^2) dx \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} -\frac{\ell-x}{\ell} \Phi|_0^\ell + \int_0^\ell \frac{10}{\ell} (\ell - x) dx \\ -\frac{x}{\ell} \Phi|_0^\ell + \int_0^\ell \frac{10}{\ell} x dx \end{Bmatrix} \quad (1.34)$$

After the evaluation of the definite integrals in the coefficient matrix and the boundary line terms in the right-hand side, Eq. (1.34) gives

$$\begin{bmatrix} \frac{\ell}{3} - \frac{1}{\ell} & \frac{1}{\ell} + \frac{\ell}{6} \\ \frac{1}{\ell} + \frac{\ell}{6} & \frac{\ell}{3} - \frac{1}{\ell} \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} q_1 + 5\ell \\ -q_2 + 5\ell \end{Bmatrix} \quad (1.35)$$

Therefore using the discretization shown in Fig. 1.5 where  $\ell = 0.25$

$$\begin{bmatrix} -\frac{11.75}{3} & \frac{24.25}{6} \\ \frac{24.25}{6} & -\frac{11.75}{3} \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} q_1 + 1.25 \\ -q_2 + 1.25 \end{Bmatrix} \quad (1.36)$$

Equation (1.36) is the common form of the elemental stiffness equation in this example.

*Step 6. Assembly of the elemental stiffness equations into a global system of algebraic equations*

Elemental stiffness equations are assembled over their common nodes to yield

$$\begin{bmatrix} -\frac{11.75}{3} & \frac{24.25}{6} & 0 & 0 & 0 \\ \frac{24.25}{6} & -\frac{23.5}{3} & \frac{24.25}{6} & 0 & 0 \\ 0 & \frac{24.25}{6} & -\frac{23.5}{3} & \frac{24.25}{6} & 0 \\ 0 & 0 & \frac{24.25}{6} & -\frac{23.5}{3} & \frac{24.25}{6} \\ 0 & 0 & 0 & \frac{24.25}{6} & -\frac{11.75}{3} \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} q_1 + 1.25 \\ 2.5 \\ 2.5 \\ 2.5 \\ -q_5 + 1.25 \end{Bmatrix} \quad (1.37)$$

Note that in equation system (1.37) the coefficients matrix is symmetric, sparse (i.e. a significant number of its members are zero) and banded. The symmetry of coefficients matrix in the global finite element equations depends on the nature of the original differential equation being solved and is not guaranteed for all cases (in particular, in most fluid flow problems this matrix will not be symmetric a). However, finite element formulations always yield sparse and banded sets of equations. This property is usually utilized to minimize computing costs in complex problems.

*Step 7. Imposition of the boundary conditions*

Prescribed values of the unknown function at the boundaries of  $\Omega$  (i.e.  $T_1 = 0$ ,  $T_5 = 1$ ) are inserted into the system of algebraic equations (1.37) and redundant equations corresponding to the boundary nodes eliminated from the set. After algebraic manipulations the following set of equations is obtained

$$\begin{bmatrix} -\frac{23.5}{3} & \frac{24.25}{6} & 0 \\ \frac{24.25}{6} & -\frac{23.5}{3} & \frac{24.25}{6} \\ 0 & \frac{24.25}{6} & -\frac{23.5}{3} \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 2.5 - \frac{24.25}{6} \times 1 \\ 2.5 \\ 2.5 - \frac{24.25}{6} \times 4 \end{Bmatrix} \quad (1.38)$$

*Step 8. Solution of the algebraic equations*

Equation set (1.38) is a determinate system and its solution gives

$$\begin{cases} T_2 = 0.95 \\ T_3 = 1.4598 \\ T_4 = 2.4979 \end{cases} \quad (1.39)$$

The analytical solution for this problem is

$$T(x) = -9 \cos(x) - 1.3515 \sin(x) + 10 \quad (1.40)$$

The comparison of the discrete solutions given in set (1.39) with their corresponding values from Eq. (1.40) reveals an acceptable degree of accuracy in the finite element results (Fig. 1.6), despite using a coarse mesh of only four elements. For example, the relative error corresponding to the node located at the middle of the domain is less than 0.1%. A further point to consider is that in this simple one-dimensional example coordinate mapping to a master element is not used.

However, as mentioned earlier, in general, a mapping from the global domain to a master element should be carried out to have a uniform limit for the integrals in the elemental matrices.

In the finite element solution of a realistic engineering problem, the global set of algebraic equations obtained after the assembly of elemental contributions will be very large (usually consisting of several thousand algebraic equations). In many cases the global equations needed to be solved are also ill-conditioned (Gerald and Wheatley, 1984). Ill-conditioned sets usually include equations in which the coefficients of unknowns range over many orders of magnitude. If procedures which cannot preserve the precision of the calculations are used, round-off errors badly affect the small

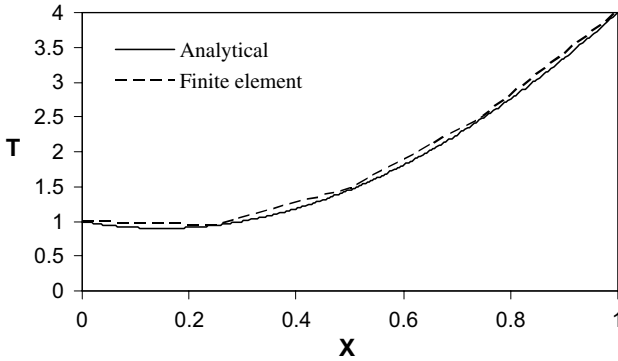


Fig. 1.6 Comparison between exact and finite element solutions.

coefficients and hence the nature of equations changes. This results in a very high degree of approximation, rendering the solutions unacceptable. Therefore, the solution of the global system of algebraic equations is regarded as one of the most important steps in the finite element modeling of realistic problems. Commonly used techniques for the solution of sets of algebraic equations in finite element schemes include “Direct” elimination methods, such as the “Frontal Solution” or “LU Decomposition,” and “Iterative” procedures, such as the “Preconditioned Conjugate Gradient Method.” Computing economy, speed, and the required accuracy of the solutions are the most important factors that should be taken into account in selecting solver routines for finite element programming. Description and full listings of computer codes based on these techniques can be found in the literature (e.g. see Hood, 1976; Hinton and Owen, 1977; Irons and Ahmad, 1980).

### 1.3.2 Example 2

To consolidate the description of the weighted residuals finite element method, we now consider the solution of the following two-dimensional Poisson equation over a rectangular domain subject to homogeneous boundary conditions (i.e.  $T = 0$  everywhere on the domain boundary).

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 2(x^2 + y^2 - x - y) \quad (1.41)$$

Starting with the usual step in the Galerkin finite element method, the problem domain is discretized into four rectangular elements as shown in Fig. 1.7.

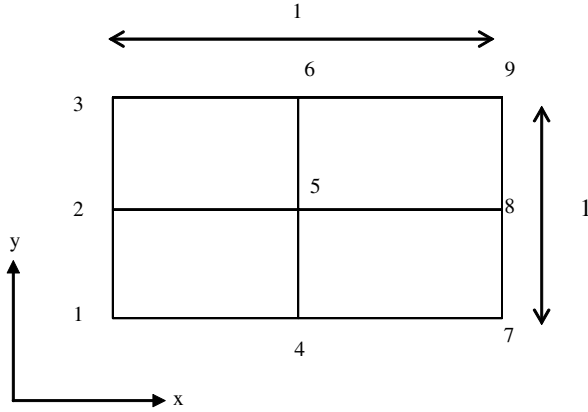


Fig. 1.7 Discretized domain showing a mesh of four bilinear elements.

Within each element the unknown function  $T$  is approximated using interpolation functions constructed as the products of linear Lagrangian interpolation functions in  $x$  and  $y$  directions. Therefore these elements are said to be four-node bilinear tensor product elements. A detailed explanation of the derivation of these functions are given in the next chapter of this book.

$$T \approx \tilde{T} = \sum_{i=1}^4 N_i(x, y) \cdot T_i \quad (1.42)$$

where  $N_i(x, y)$  are the interpolation functions associated with nodes  $i = 1, \dots, 4$ . The weighted residual statement of the problem is therefore written as

$$\iint_{\Omega_e} w \left( \frac{\partial^2 \tilde{T}}{\partial x^2} + \frac{\partial^2 \tilde{T}}{\partial y^2} - 2(x^2 + y^2 - x - y) \right) dx dy = 0 \quad (1.43)$$

In the standard Galerkin method, weight functions are identical to the shape functions and hence

$$\iint_{\Omega_e} [N]^T \left( \frac{\partial^2 \tilde{T}}{\partial x^2} + \frac{\partial^2 \tilde{T}}{\partial y^2} - 2x^2 - 2y^2 + 2x + 2y \right) dx dy = 0 \quad (1.44)$$

For simplicity of writing we write Eq. (1.44) as

$$\int_A [N]^T \left( \left( \frac{\partial^2 \tilde{T}}{\partial x^2} + \frac{\partial^2 \tilde{T}}{\partial y^2} - 2(x^2 + y^2 - x - y) \right) \right) dA = 0 \quad (1.45)$$

After the application of Green's theorem second-order derivatives disappear and we have

$$\int_{\Gamma} [N]^T \left( \frac{\partial \tilde{T}}{\partial x} \cos \theta + \frac{\partial \tilde{T}}{\partial y} \sin \theta \right) d\Gamma - \int_A \left( \frac{\partial [N]^T}{\partial x} \frac{\partial \tilde{T}}{\partial x} + \frac{\partial [N]^T}{\partial y} \frac{\partial \tilde{T}}{\partial y} \right) dA - \int_A [N]^T (2(x^2 + y^2 - x - y)) dA = 0 \tag{1.46}$$

where  $\Gamma$  is the element boundary line and  $\theta$  is the angle between outward normal vector to the boundary and  $\Gamma$ . The weighted residual statement (1.46) consists of the following three components

$$[k^{(e)}] = \left( \int_A \left( \frac{\partial [N]^T}{\partial x} \frac{\partial \tilde{T}}{\partial x} + \frac{\partial [N]^T}{\partial y} \frac{\partial \tilde{T}}{\partial y} \right) \right) dA \tag{1.47}$$

$$\{f^{(e)}\} = \int_A [N]^T (2(x^2 + y^2 - x - y)) dA \tag{1.48}$$

$$\{I^{(e)}\} = \int_{\Gamma} [N]^T \left( \frac{\partial \tilde{T}}{\partial x} \cos \theta + \frac{\partial \tilde{T}}{\partial y} \sin \theta \right) d\Gamma \tag{1.49}$$

We now calculate each of these terms using the previously described mapping onto a master element with its associated local coordinate system shown in Fig. 1.8

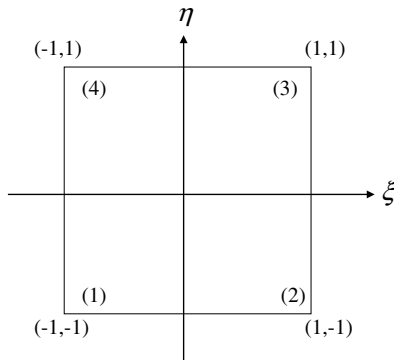


Fig. 1.8 Four-node square element and its associated local coordinate system  $(\xi, \eta)$ .

$$[k^{(e)}] = \frac{1}{16} \int_{-1}^{+1} \int_{-1}^{+1} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} & \frac{\partial N_i}{\partial \eta} \\ \frac{\partial N_j}{\partial \xi} & \frac{\partial N_j}{\partial \eta} \\ \frac{\partial N_k}{\partial \xi} & \frac{\partial N_k}{\partial \eta} \\ \frac{\partial N_l}{\partial \xi} & \frac{\partial N_l}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} & \frac{\partial N_j}{\partial \xi} & \frac{\partial N_k}{\partial \xi} & \frac{\partial N_l}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} & \frac{\partial N_j}{\partial \eta} & \frac{\partial N_k}{\partial \eta} & \frac{\partial N_l}{\partial \eta} \end{bmatrix} d\xi d\eta \quad (1.50)$$

where  $i, j, k$ , and  $l$  are node indices. Substitution for the derivatives of the shape functions in Eq. (1.50) gives

$$[k^{(e)}] = \frac{1}{16} \int_{-1}^{+1} \int_{-1}^{+1} \begin{bmatrix} -(1-\eta) & -(1-\xi) \\ (1-\eta) & -(1+\xi) \\ (1+\eta) & (1+\xi) \\ -(1+\eta) & (1-\xi) \end{bmatrix} \begin{bmatrix} (1-\eta) & (1-\eta) & (1+\eta) & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & (1+\xi) & (1-\xi) \end{bmatrix} d\xi d\eta \quad (1.51)$$

and

$$[k^{(e)}] = \frac{1}{16} \int_{-1}^{+1} \int_{-1}^{+1} \begin{bmatrix} (1-\eta)^2 + (1-\xi)^2 & (1-\xi^2) - (1-\eta)^2 \\ (1-\xi^2) - (1-\eta)^2 & (1-\eta)^2 + (1+\xi)^2 \\ -(1-\eta^2) - (1-\xi^2) & (1-\eta^2) - (1+\xi)^2 \\ (1-\eta^2) - (1-\xi)^2 & -(1-\eta)^2 - (1-\xi^2) \\ -(1-\eta^2) - (1-\xi^2) & (1-\eta^2) - (1-\xi^2)^2 \\ (1-\eta^2) - (1+\xi)^2 & -(1-\eta^2) - (1-\xi^2) \\ (1+\eta)^2 + (1+\xi)^2 & (1-\xi^2) - (1+\eta)^2 \\ (1-\xi^2) - (1+\eta)^2 & (1+\eta)^2 + (1-\xi^2)^2 \end{bmatrix} d\xi \quad (1.52)$$

After the evaluation of the integrals, Eq. (1.52) gives

$$[k^{(e)}] = \begin{bmatrix} \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{bmatrix} \begin{matrix} i \\ j \\ k \\ l \end{matrix} \quad (1.53)$$

Similarly

$$\{f^{(e)}\} = \int_A [N]^T \left( 2 \left( \frac{(1+\xi)^2}{16} + \frac{(1+\eta)^2}{16} - \frac{(1+\xi)}{4} - \frac{(1+\eta)}{4} \right) \right) dA \tag{1.54}$$

Hence we have

$$\{f^{(e)}\} = \frac{1}{32} \int_{-1}^{+1} \int_{-1}^{+1} \left( \frac{(1+\xi)^2}{16} + \frac{(1+\eta)^2}{16} - \frac{(1+\xi)}{4} - \frac{(1+\eta)}{4} \right) \begin{bmatrix} (1-\xi)(1-\eta) \\ (1+\xi)(1-\eta) \\ (1+\xi)(1+\eta) \\ (1-\xi)(1+\eta) \end{bmatrix} d\xi d\eta \tag{1.55}$$

or

$$\{f^{(e)}\} = \frac{1}{32} \int_{-1}^{+1} \int_{-1}^{+1} \begin{bmatrix} \frac{(1+\xi)^2(1-\xi)(1-\eta)}{16} + \frac{(1+\eta)^2(1-\eta)(1-\xi)}{16} - \frac{(1-\xi)^2(1-\eta)}{4} - \frac{(1-\eta)^2(1-\xi)}{4} \\ \frac{(1+\xi)^3(1-\eta)}{16} + \frac{(1+\eta)^2(1-\eta)(1+\xi)}{16} - \frac{(1+\xi)^2(1-\eta)}{4} - \frac{(1-\eta)^2(1+\xi)}{4} \\ \frac{(1+\xi)^3(1+\eta)}{16} + \frac{(1+\eta)^2(1+\xi)}{16} - \frac{(1+\xi)^2(1+\eta)}{4} - \frac{(1+\eta)^2(1+\xi)}{4} \\ \frac{(1+\xi)^2(1-\xi)(1-\eta)}{16} + \frac{(1+\eta)^2(1-\eta)(1-\xi)}{16} - \frac{(1-\xi)^2(1-\eta)}{4} - \frac{(1-\eta)^2(1-\xi)}{4} \end{bmatrix} d\xi d\eta \tag{1.56}$$

and finally

$$\{f^{(e)}\} = \begin{bmatrix} -\frac{1}{32} \\ -\frac{1}{24} \\ -\frac{5}{96} \\ -\frac{1}{24} \end{bmatrix} \begin{matrix} i \\ j \\ k \\ l \end{matrix} \tag{1.57}$$

The flux term along the inner boundaries between the elements of the domain cancel each other out during the assembly of the elemental stiffness equations. As in the exterior boundaries, essential boundary conditions (i.e. function values) are given there is no need to solve any equations. Therefore there is no need to calculate the third component of Eq. (1.46). Assembling the elemental equations we obtain the following global stiffness equation and load vectors (i.e. R.H.S.) for the present example.

Global stiffness equation

$$\begin{bmatrix}
 \overset{1}{0.66670} & \overset{2}{-0.1667} & \overset{3}{0} & \overset{4}{-0.1667} & \overset{5}{-0.3333} & \overset{6}{0} \\
 -0.1667 & 1.33330 & -0.1667 & -0.3333 & -0.3333 & -0.3333 \\
 0 & -0.1667 & 0.66670 & 0 & -0.3333 & -0.1667 \\
 -0.1667 & -0.3333 & 0 & 1.33330 & -0.3333 & 0 \\
 -0.3333 & -0.3333 & -0.3333 & -0.3333 & 2.66670 & -0.3333 \\
 0 & -0.3333 & -0.1667 & 0 & -0.3333 & 1.33330 \\
 0 & 0 & 0 & -0.1667 & -0.3333 & 0 \\
 0 & 0 & 0 & -0.3333 & -0.3333 & -0.3333 \\
 0 & 0 & 0 & 0 & -0.3333 & -0.1667 \\
 & & & \overset{7}{0} & \overset{8}{0} & \overset{9}{0} \\
 & & & 0 & 0 & 0 \\
 & & & 0 & 0 & 0 \\
 & & & -0.1667 & -0.3333 & 0 \\
 & & & -0.3333 & -0.3333 & -0.3333 \\
 & & & 0 & -0.3333 & -0.1667 \\
 & & & 0.66670 & -0.1667 & 0 \\
 & & & -0.1667 & 1.33330 & -0.1667 \\
 & & & 0 & -0.1667 & 0.66670
 \end{bmatrix} \quad (1.58)$$

Load vector

$$\begin{bmatrix}
 -0.0313 \\
 -0.0729 \\
 -0.0417 \\
 -0.0729 \\
 -0.1667 \\
 -0.0938 \\
 -0.0417 \\
 -0.0938 \\
 -0.0521
 \end{bmatrix} \cdot \quad (1.59)$$

After the imposition of the boundary conditions and simplifying the global system of equations we obtain

$$[2.66670][T_5] = 0.1667 \quad (1.60)$$

and

$$T_5 = 6.25e^{-02} \quad (1.61)$$

The analytical solution of the original equation is

$$T = x(1 - x) \quad y(1 - y) \quad (1.62)$$

Insertion of the coordinates at the centre of the domain gives the exact result for this point again as  $T = 6.25e^{-02}$ . Therefore despite using a very coarse mesh the application of standard Galerkin method in this example yields a super-convergent result.

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