# Preface

Consider the following list of problems:

- Finding the global minimum of a function on a subset of $\mathbb{R}^n$,
- Solving equations,
- Computing the Lebesgue volume of a subset $\mathbf{S} \subset \mathbb{R}^n$,
- Computing an upper bound on $\mu(\mathbf{S})$ over all measures $\mu$ satisfying some moment conditions,
- Pricing exotic options in Mathematical Finance,
- Computing the optimal value of an optimal control problem,
- Evaluating an ergodic criterion associated with a Markov chain,
- Evaluating a class of multivariate integrals,
- Computing Nash equilibria,
- With $\widehat{f}$ the convex envelope of a function $f$, evaluate $\widehat{f}(\mathbf{x})$ at some given point $\mathbf{x}$.

The above seemingly different and unrelated problems share in fact a very important property: they all can be viewed as a particular instance of the *Generalized Moment Problem* (GMP in short)! And of course the above list is not exhaustive!

It is known that the GMP has great modeling power with impact in several branches of Mathematics and also with important applications in various fields. However, and as illustrated by the above list, in its full generality the GMP cannot be solved numerically. According to Diaconis (1987), "the theory [of moment problems] is not up to the demands of applications".

One invoked reason is the high complexity of the problem: "numerical determination ... is feasible for a small number of moments, but appears to be quite difficult in general cases", whereas Kemperman (1987) points out the lack of a general algorithmic approach. Indeed quoting Kemperman:

"...a deep study of algorithms has been rare so far in the theory of moments, except for certain very specific practical applications, for instance, to crystallography, chemistry and tomography. No doubt, there is a considerable need for developing reasonably good numerical procedures for handling the great variety of moment problems which do arise in pure and applied mathematics and in the sciences in general...".

The main purpose of this book is to show that the situation becomes much nicer for the GMP with polynomial data (and sometimes one may even consider GMPs with some piecewise polynomial data or even rational functions). Indeed, results from real algebraic geometry and functional analysis have provided new characterizations of polynomials positive on a basic semi-algebraic set $\mathbb{K} \subset \mathbb{R}^n$, and *dual* results on moment sequences that can be represented by finite Borel measures supported on $\mathbb{K}$. This beautiful duality is nicely captured by standard duality in convex analysis, applied to some appropriate convex cones. Moreover, these characterizations are not only very elegant and simple to state, but more importantly, it also turns out that they are amenable to practical computation as they can be checked by semidefinite programming (and sometimes linear programming), two well known powerful techniques of convex optimization.

Conjunction of the above theoretical breakthrough with the development of semidefinite programming has allowed to define a numerical scheme based on semidefinite programming to approximate, and sometimes solve exactly, the original GMP. It merely consists of a hierarchy of semidefinite relaxations of the GMP where each semidefinite relaxation is a convex optimization problem for which efficient public softwares are available. As shown in the book, the beautiful duality between moments and positive polynomials is perfectly expressed by standard duality in convex optimization, when applied to these semidefinite relaxations.

This book is an attempt to convince the reader that one may now consider solving (or at least approximating as closely as desired) some difficult problems that were thought to be out of reach a few years ago, or for which only heuristics were available. Of course, since such problems remain hard, this methodology has some practical limitations mainly due to the size of the original problem, especially in view of the present status of semidefinite programming solvers (used as a black box subroutine to solve each semidefinite relaxation). However, we also indicate how sparsity or symmetries (when present) can be exploited so as to handle problems of larger size. Hence much remains to be done but we hope that this methodology will open the door to a more systematic and efficient treatment of such problems.

We introduce this methodology in a unified manner. To do so we first provide a general numerical scheme for solving (or approximately solving) the abstract GMP with polynomial data. Convergence and several other properties are proved in a very general context. Next, we illustrate in detail the above general methodology when applied to several instances of the GMP in various and very different applications in Global Optimization, Algebra, Probability and Markov Chains, Optimal Control, Mathematical Finance, Multivariate Integration, etc..., possibly with *ad hoc* adjustments if and when necessary. Depending on the problem on hand, additional insights are also provided.

The book is divided into two main parts, and here is a brief chapter-by-chapter description of its content. Part I is devoted to the theoretical basis that supports the proposed methodology to solve or approximate the abstract GMP. Part II is devoted to illustrate (and sometimes complement) the methodology for specific instances of the GMP in various domains.

**Part I**

**Chapter 1** describes the abstract basic GMP and its dual with a few examples. We also provide some general results concerning the GMP with polynomial data and its dual, and show why the theory of moments and its dual theory of positive polynomials can be useful to solve the GMP.
**Chapter 2** reviews basic results of real algebraic geometry on the representation of positive polynomials, among which the fundamental Positivstellensatze of Krivine, Stengle, Schmüdgen, Putinar and Jacobi and Prestel. We also provide some additional representation results that take account of several specific cases like for instance, finite varieties, convex semi-algebraic sets, representations that preserve sparsity, etc.
**Chapter 3** is the *dual* of Chapter 2 as most results are the *dual* analogues of those described in Chapter 2. Indeed the problem of representing polynomials that are positive on a set $\mathbb{K}$ has a dual facet which is the problem of characterizing sequences of reals that are moment sequences of some finite Borel measure supported on $\mathbb{K}$. Moreover, as we shall see, this beautiful duality is nicely captured by standard duality in convex analysis, applied to some appropriate convex cones. We review basic results in the moment problem and also particularize to some specific important cases like those

in Chapter 2.
**Chapter 4**, now with the appropriate theoretical tools on hand, describes the general methodology to solve the abstract GMP. That is, we provide a hierarchy of semidefinite (or sometimes linear) relaxations of the basic GMP, whose associated sequence of optimal values converges to the optimal value of the GMP. Variants of this methodology are also provided to handle additional features such as countably many moment constraints, several measures, or GMP with sparsity properties. This chapter should allow the reader to see what is the basic idea underlying the methodology that permits to solve (or approximate) a problem formulated as a particular instance of the GMP.

## Part II

Part II of the book is devoted to convince the reader about the power of the moment approach for solving the generalized moment problem with polynomial data. Each of the next chapters illustrates (and sometimes complements) the above methodology on some particular important instances of the GMP. For each chapter we detail the semidefinite relaxations of Chapter 4 in the specific context of the chapter. Moreover, depending on the specificity of the problem on hand, additional insights are also provided.
**Chapter 5** is about global optimization, probably the simplest instance of the generalized moment problem. We detail the semidefinite relaxations of Chapter 4 for minimizing a polynomial on $\mathbb{R}^n$ and on a compact basic semi-algebraic set $\mathbb{K} \subset \mathbb{R}^n$. We also discuss the linear relaxations and several particular cases, e.g. when $\mathbb{K}$ is a polytope or a finite variety. In particular, this latter case encompasses all $0-1$ discrete optimization problems.
**Chapter 6** is about solving systems of polynomial equations. Of course, if the goal is to search for just one solution, one may minimize any polynomial criterion and see the problem as a particular case of Chapter 5. But we also consider the case where one searches for *all* complex or *all* real solutions and show that the moment approach is well-suited to solve this problem and provides the first algorithm to compute all real solutions without computing all the complex solutions, in contrast with the usual algebraic approaches based on Gröbner bases or homotopy.
**Chapter 7** covers some applications in probability. We first consider the problem of computing an upper bound on $\mu(\mathbf{S})$ for some subset $\mathbf{S} \subset \mathbb{R}^n$, over all measures $\mu$ that satisfy certain moment conditions. We then consider

the difficult problem of computing (or at least approximating) the volume of a compact basic semi-algebraic set. We end up with the mass-transfer (or Monge-Kantorovich) problem.

**Chapter 8** is about Markov chains and invariant probabilities. We first address the problem of computing an upper bound on $\mu(\mathbf{S})$ for some $\mathbf{S} \subset \mathbb{R}^n$, over all invariant probability measures $\mu$ of a given Markov chain on $\mathbb{R}^n$. We then consider the problem of approximating the value of an ergodic criterion, as an alternative to simulation which only provides a random estimate.

**Chapter 9** considers an important application in mathematical finance, namely the pricing of exotic options under a no-arbitrage assumption, first with only knowledge of some moments of the distribution of the underlying asset price, and then when one assumes that the asset price obeys some Ito stochastic differential equation.

**Chapter 10** considers an application in control. We apply the moment approach to the so-called weak formulation of optimal control problems in which the initial problem is viewed as an infinite linear-programming model over suitable *occupation measures*, an instance of the generalized moment problem.

**Chapter 11** considers the following problem: Given a rational function $f$ on a basic semi-algebraic set $\mathbb{K}$, evaluate $\widehat{f}(\mathbf{x})$ at a particular point $\mathbf{x}$ in the domain of the convex envelope $\widehat{f}$ of $f$. We then consider semidefinite representations for the convex hull $\mathrm{co}(\mathbb{K})$ of $\mathbb{K}$. That is, finding a set defined by linear matrix inequalities in a lifted space such that $\mathrm{co}(\mathbb{K})$ is a suitable projection of that set.

**Chapter 12** is about approximating the multivariate integral of a rational function or an exponential of a multivariate polynomial. We then consider the moment approach as a tool for evaluating gradients or Hesssians in the maximum entropy approach for estimating an unknown density based on knowledge of some of its moments.

**Chapter 13** first considers the problem of minimizing the supremum of finitely many rational functions on a basic semi-algebraic set. Then this is used to compute (or approximate) the value of Nash equlibria for N-player finite games. We end up with applying the moment approach to 2-player zero-sum polynomial games.

**Chapter 14**, our final chapter, applies the moment approach to provide bounds on functionals of the solution of linear partial differential equations with boundary conditions and polynomial coefficients.

• The applications of the GMP described in this book come from various areas, e.g., Optimal Control, Optimization, Mathematical Finance, Probability and Operations Research. Therefore, in the Appendices at the end of the book, we have provided some brief basic background on results from optimization and probability that are used in some of the chapters.

• At some places in the book, some theorems, lemmas or propositions are framed into a box to emphasize their importance, at least to the author's taste.

• Sometimes, proofs of theorems, lemmas and propositions are postponed for the sake of clarity of exposition and for the reader to avoid from being lost in technical details in the middle of a chapter. Sometimes, if short and/or important, a proof is provided just after the corresponding theorem, lemma or proposition. Finally, if it is too long, or too technical, or not crucial, a proof is simply omitted but a reference is provided in the *Notes and Sources* section at the end of the corresponding chapter.