

Chapter 1

Orders of Magnitude and Dimensional Analysis

1.1 A few orders of magnitude *

When one solves a physics problem, it is important to rely on intuition, for instance to be able to guess the parameters on which an unknown quantity depends, or to estimate, before any calculation, the order of magnitude of the numerical value of the result. For that, it is necessary to know the orders of magnitudes of physical quantities that are characteristic of typical systems. This allows in particular to detect immediately obvious errors when one calculates a numerical value.

In the following exercise, one thus asks, for the most common physical quantities (lengths, masses, times, velocities...), the (approximate) numerical values corresponding to a few representative systems.

Universal constants

- (1) What is the numerical value of the gravitational constant G ?
- (2) What is the numerical value of Coulomb's constant $1/4\pi\epsilon_0$?
- (3) What is the numerical value of the permeability of vacuum μ_0 ?
- (4) What is the numerical value of Planck's constant h ?
- (5) What is the numerical value of Boltzmann's constant k_B ?
- (6) What is the numerical value of the ideal gas constant R ?
- (7) What is the numerical value of Avogadro's number \mathcal{N}_A ?

Lengths

- (1) What is the typical size of an atomic nucleus?
- (2) What is the typical size of an atom?
- (3) To which range of wavelengths does the visible spectrum correspond?
- (4) What is the value of the Earth's radius?

- (5) What is the distance between the Earth and the Moon?
- (6) What is the distance between the Earth and the Sun?

Masses

- (1) What is the mass of an electron?
- (2) What is the mass of a proton? And that of a neutron?
- (3) What is the mass of the Earth?
- (4) What is the mass of the Sun?

Densities

- (1) What is the density of water?
- (2) What is order of magnitude of the density of a usual metal?
- (3) What is the density of air at the standard conditions for temperature and pressure?

Times

- (1) What is the duration of a day in seconds?
- (2) What is the duration of a year in seconds?

Frequencies

- (1) What is the frequency range of electromagnetic waves corresponding to visible light?
- (2) What is the frequency of electromagnetic waves used in a microwave oven?
- (3) What is the frequency range used in radio broadcasting?
- (4) To which frequency range do sound waves belong?

Velocities

- (1) What is the typical speed of a satellite orbiting on a low-lying orbit around the Earth?
- (2) What is the speed of sound in air?
- (3) What is the speed of light in vacuum?

Energies

- (1) What is the typical order of magnitude of a molecular binding energy?
- (2) What is the typical energy of an atom in an ideal monoatomic gas at temperature T ?

- (3) What is the energy required to increase the temperature of one gram of water by one degree Celsius?

Powers

- (1) What is the typical power of a light bulb?
 (2) What is the typical power of a car engine?
 (3) What is the typical power of a nuclear power plant? And that of a wind turbine?
 (4) What is the average power consumed by the human body?

Solution

Universal constants

- (1) The gravitational constant G is about $6.67 \times 10^{-11} \text{ m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2}$.
 (2) $1/4\pi\epsilon_0 \simeq 9 \times 10^9 \text{ m} \cdot \text{F}^{-1}$.
 (3) By definition $\mu_0 = 4\pi \times 10^{-7} \text{ H} \cdot \text{m}^{-1}$. Note that one has the *exact* relationship $\mu_0\epsilon_0c^2 = 1$. Since μ_0 and c have an *exact* value, so has ϵ_0 .
 (4) Planck's constant is approximately $6.63 \times 10^{-34} \text{ J} \cdot \text{s}$. One often uses the *reduced Planck's constant* $\hbar \equiv h/2\pi \simeq 1.05 \times 10^{-34} \text{ J} \cdot \text{s}$.
 (5) Boltzmann's constant k_B is approximately equal to $1.38 \times 10^{-23} \text{ J} \cdot \text{K}^{-1}$.
 (6) The ideal gas constant has the numerical value $R \simeq 8.314 \text{ J} \cdot \text{K}^{-1} \cdot \text{mol}^{-1}$.
 (7) One has $\mathcal{N}_A \simeq 6.022 \times 10^{23} \text{ mol}^{-1}$. Note that the ideal gas constant is related to Boltzmann's constant and to Avogadro's number by $R = \mathcal{N}_A k_B$.

Lengths

- (1) The typical size of an atomic nucleus is on the order of one femtometer ($1 \text{ fm} = 10^{-15} \text{ m}$).
 (2) The typical size of an atom is on the order of 10^{-10} m , i.e. an Angstrom (\AA). More precisely, the Bohr radius of hydrogen is $a_0 = 0.53 \text{ \AA}$.
 (3) The visible spectrum extends from the violet to the red, i.e. from 400 to 800 nm.
 (4) The Earth's radius is about 6,400 km, implying a circumference of 40,000 km (the first definition of the meter, dating back to the French Revolution, defined the meter as "the ten-millionth part of one quarter of the Earth meridian".)

- (5) The distance between the Earth and the Moon is about 384,000 km, i.e. a little bit more than a light-second.
- (6) The distance between the Earth and the Sun (which defines the *astronomical unit*) is about 150 million kilometers (or 8 light-minutes).

Masses

- (1) The mass of an electron is $m_e \simeq 9.1 \times 10^{-31}$ kg. One can also remember its value expressed in energy units (*via* the relationship $E = mc^2$), which is 511 keV (we recall that $1 \text{ eV} = 1.6 \times 10^{-19}$ J).
- (2) The neutron and the proton have almost the same mass, on the order of 1.67×10^{-27} kg, or 938 MeV. The proton is about 1,836 times as massive as the electron. The neutron is in fact very slightly more massive than the proton, which makes the β decay of a free neutron (into a proton, an electron and an antineutrino) energetically possible.
- (3) The mass of the Earth is about 6×10^{24} kg.
- (4) The mass of the Sun is about 2×10^{30} kg.

Densities

- (1) The density of water is $1,000 \text{ kg} \cdot \text{m}^{-3}$.
- (2) The density of iron, for example, is $7,900 \text{ kg} \cdot \text{m}^{-3}$; the density of copper $8,900 \text{ kg} \cdot \text{m}^{-3}$. Some metals, like the alkalis, have a very small density (like lithium, $530 \text{ kg} \cdot \text{m}^{-3}$). The density of mercury is $13,530 \text{ kg} \cdot \text{m}^{-3}$; the density of platinum (used, together with iridium, for the realization of the kilogram prototype) is $21,450 \text{ kg} \cdot \text{m}^{-3}$.
- (3) The density of air at the standard conditions for temperature and pressure (273 K and 10^5 Pa) is

$$\rho = \frac{PM}{RT} \simeq 1.3 \text{ kg} \cdot \text{m}^{-3}.$$

Times

- (1) A *solar* day (24 hours) has a duration of $24 \times 3,600 = 86,400$ s. The *sidereal* day has a duration of 86,164 s.
- (2) A year is 365.25 days, or about 30 million seconds.

Frequencies

- (1) The visible domain corresponds to frequencies ranging from 4×10^{14} Hz (red light) to 8×10^{14} Hz (violet light).
- (2) A microwave oven uses microwaves at 2.45 GHz.

- (3) For radio broadcasting, one uses waves with frequencies around 100 kHz (for amplitude modulation) or 100 MHz (for frequency modulation).
- (4) The human ear is sensitive to sound waves of frequencies ranging from 20 Hz to 20 kHz.

Velocities

- (1) A satellite on a low orbit around the Earth has a velocity on the order of $\sqrt{GM_T/R_T}$, with M_T the mass of the Earth and R_T its radius. One finds a speed of about $8 \text{ km} \cdot \text{s}^{-1}$. It thus takes about $40,000/8 \simeq 5,000 \text{ s}$ (a little bit more than one hour) for the satellite to go round the Earth.
- (2) Sound waves need a medium to propagate. The speed of sound in air, at ambient temperature, is about $340 \text{ m} \cdot \text{s}^{-1}$. For a gas of molar mass M , at temperature T , one has $c_{\text{sound}} = \sqrt{\gamma RT/M}$, where R is the ideal gas constant and γ the ratio of specific heats at constant pressure and at constant volume ($\gamma = 1.4$ for a gas of diatomic molecules such as air). In water, the speed of sound is about $1.5 \text{ km} \cdot \text{s}^{-1}$.
- (3) The speed of light in a vacuum has (by definition of the meter) the fixed value $299,792,458 \text{ m} \cdot \text{s}^{-1}$. One uses in general $c \simeq 3 \times 10^8 \text{ m} \cdot \text{s}^{-1}$. In a transparent medium with refractive index n , the velocity of light is c/n .

Energies

- (1) Molecular binding energies are on the order of a few electronvolts ($1 \text{ eV} = 1.6 \times 10^{-19} \text{ J}$).
- (2) The energy of an atom in a monoatomic ideal gas at temperature T is $3k_B T/2$, i.e. for $T = 300 \text{ K}$, on the order of $6 \times 10^{-21} \text{ J}$.
- (3) The energy required to increase, by one degree Celsius, the temperature of one gram of water is what defines the *calorie*, and is equivalent to 4.18 J .

Powers

- (1) A light bulb has a typical power of a few tens of watts.
- (2) Even today, one measures the power of car engines in *horsepower*, which corresponds to 735 W . A 100 hp car therefore has a power of 74 kW .
- (3) A nuclear power plant has a power ranging from 100 to 1,000 MW. A wind turbine has a typical power of 1 MW .
- (4) It is well known that a human being must consume about 2,000 kilocalories per day. As one kilocalorie corresponds to 4.18 kJ , the average

power dissipated by a human being is

$$P \simeq \frac{2 \times 10^3 \times 4.18 \times 10^3}{24 \times 3,600} \simeq 100 \text{ W.}$$

The instantaneous power can reach several kW, e.g. when performing physical effort.

1.2 Dimensional analysis *

In this exercise, one applies, to a few simple examples, the technique of dimensional analysis, which allows one to obtain, without any calculation, the form of the solution of a physics problem, by combining the relevant physical parameters of the problem into quantities that have the same physical dimension (length, mass, time...) as the results¹. In cases where the number of parameters is small, one often encounters the situation in which only one combination of the parameters has the right dimension. One thus obtains the correct result within a (dimensionless) numerical factor, which is often close to unity. For numerical calculations, one will use the values recalled in the solution of Problem 1.1.

Free fall. A particle of mass m falls from a height ℓ , without initial velocity, in the gravity field g . How does the fall time scale with ℓ and g ?

Classical radius of the electron. One assumes that the electron (mass m_e , charge q_e) can be described by a spherical charge distribution, with a radius r_e . In the framework of classical (i.e. non-quantum) relativistic theory, calculate r_e . What is the numerical value of r_e ? Comment.

Bohr radius of the hydrogen atom. Can one find, by dimensional analysis, the typical size of the orbit of an electron around a proton, in the framework of classical mechanics? Why? Show that introducing, in quantum mechanics, the reduced Planck's constant \hbar allows one to obtain the characteristic size of the orbit (hint: \hbar has the dimension of an *action*, i.e. of an energy multiplied by a time, or, equivalently, of an angular momentum).

¹It is possible to make dimensional analysis formal by using the so-called “ π -theorem” : see e.g. E. Buckingham, *On physically similar systems: Illustrations of the use of dimensional analysis*, Phys. Rev. **4**, 345 (1914).



Fig. 1.1 Explosion of an American nuclear bomb in 1953. The time indicated above each image corresponds to the time elapsed since the explosion. The scale is the same for all photographs: $4 \text{ km} \times 2.5 \text{ km}$.

Plasma frequency. One considers a gas of electrons (with number density n) moving in a background of static ions, in such a way that the system is neutral. Find by dimensional analysis the expression of the characteristic oscillation frequency of this plasma.

Stokes' formula. Find by dimensional analysis the form of the drag force exerted by a fluid with viscosity η and density ρ on a sphere of radius R , moving with velocity v in this fluid. Hint: the coefficient of (dynamical) viscosity η has the dimension of a pressure divided by a (spatial) gradient of velocity. Is there *a priori* only one solution? To go further, one will assume that the force is proportional to the velocity.

Energy released in a nuclear explosion. In the fifties, the energy released in a nuclear explosion was classified information. However, movies showing the expansion of the famous “nuclear mushroom” were released to the public by the US Army. The British physicist G.I. Taylor then deduced the energy released by the explosion, by assuming that the radius R of the cloud depended only on the time t elapsed since the explosion, on the released energy E , and on the density ρ of the air around the cloud. Show that, under those assumptions, $R \sim t^\beta$. What is the value of β ? Show that by measuring $R(t)$ one can deduce E . Figure 1.1 presents a series of photographs of a thermonuclear explosion, taken for various t . Is the law $R(t)$ obtained by dimensional analysis in agreement with this series of images?

Solution

In all that follows, one denotes by $[A]$ the dimension of the physical quantity A . We recall that the three fundamental dimensions in mechanics are the length, noted L , the mass, noted M , and the time, noted T . Thus the

equation $[v] = L \cdot T^{-1}$ means that the velocity v has the dimension of a length divided by a time.

Free fall. One looks for an expression of the form $t \sim m^\alpha g^\beta \ell^\gamma$ for the free fall time, which implies, dimensionally:

$$T = M^\alpha (L \cdot T^{-2})^\beta L^\gamma$$

or, identifying the various exponents of M , L and T , $\alpha = 0$, $\beta + \gamma = 0$, and $-2\beta = 1$. This set of equations has the solution $(\alpha, \beta, \gamma) = (0, -1/2, 1/2)$, and the free fall time thus scales as $t \sim \sqrt{\ell/g}$, which is the well-known result, within a factor $\sqrt{2}$.

Classical radius of the electron. In the framework of classical electrodynamics, the only parameters that can enter the expression of r_e are the electron charge q_e , or in fact the combination $q_e^2/(4\pi\epsilon_0)$, which has the dimension of an energy multiplied by a length (ML^3T^{-2}), the electron mass m_e , and the speed of light c . One finds

$$r_e = \frac{q_e^2}{4\pi\epsilon_0 m_e c^2} \simeq 2.82 \times 10^{-15} \text{ m.}$$

The classical radius of the electron thus has the same order of magnitude as the size of the atomic nucleus; however, contrary to nuclei, electrons are, as far as is known today, *point-like elementary particles*. The classical radius of the electron only gives the typical size an electron would have *if* one could describe it classically. Quantum mechanics substitutes to this notion of radius the concept of a probability density over a distance deduced from quantum electrodynamics.

Bohr radius of the hydrogen atom. In the framework of classical, non-relativistic mechanics, the only parameters that can play a role in the expression of the orbit radius are $q_e^2/(4\pi\epsilon_0) \equiv e^2$ (see previous question), and the electron mass m_e . Since $[e^2] = ML^3T^{-2}$ and $[m_e] = M$, we have no way of eliminating the time appearing in $[e^2]$. One thus cannot, simply by dimensional analysis, obtain the radius of the electron orbit in the hydrogen atom. In fact, one expects this result, since it is well known for the Kepler problem in the case of gravitational interactions (having the same $1/r$ dependence as the Coulomb interaction) that the orbits of the planets around the Sun, for instance, have an arbitrary size, fixed by initial conditions. In that case, there is no characteristic length scale, and the

same laws explain the motion of satellites around the Earth, of planets around the Sun and of the Sun around the galactic center².

In quantum mechanics, Planck's constant \hbar , which is an *action* with dimension ML^2T^{-1} (linear momentum \times length, or, equivalently, energy \times time), allows one to fix the size a_0 of the hydrogen atom. Indeed, by writing

$$a_0 = (e^2)^\alpha \hbar^\beta m_e^\gamma$$

one must have, in order to obtain a length,

$$\begin{cases} \alpha + \beta + \gamma = 0 \\ 3\alpha + 2\beta = 1 \\ -2\alpha - \beta = 0 \end{cases}$$

with solution $(\alpha, \beta, \gamma) = (-1, 2, -1)$. One thus finds, for the expression of the size of the electron orbit in the hydrogen atom, the following value:

$$a_0 = \frac{\hbar^2}{m_e e^2} = \frac{4\pi\epsilon_0 \hbar^2}{m_e q_e^2}.$$

This quantity is called the *Bohr radius*, after the Danish physicist Niels Bohr, who devised the first quantum-mechanical model (in fact, a semi-classical one) of the atom. Numerically, one has $a_0 \simeq 0.53 \times 10^{-10}$ m.

♦ **Remark.** Using e^2 , \hbar and c one can build a dimensionless number, the *fine structure constant*, denoted α , equal to

$$\alpha \equiv \frac{e^2}{\hbar c} \simeq \frac{1}{137.04}.$$

It measures the strength of electromagnetic interactions. Note that the Bohr radius can be expressed, as a function of the classical radius of the electron, as:

$$a_0 = \frac{r_e}{\alpha^2}.$$

Plasma frequency. This frequency can only depend on q_e^2/ϵ_0 , on the density n , and on the electron mass m_e . One thus searches an expression of the form:

$$\omega_p = n^\alpha \left(\frac{q_e^2}{\epsilon_0} \right)^\beta m_e^\gamma.$$

²Let us stress, however, a fundamental difference between gravitational and Coulomb interactions: in the latter case, the existence of positive *and* negative charges implies that the long-range character of the interaction is often “masked” by screening of a charge by other charges of opposite sign. On the contrary, mass being always positive, the gravitational interaction is never screened; this is why it plays a crucial role on astronomical scales, although its strength is very weak compared to that of Coulomb interaction.

This implies, dimensionally:

$$T^{-1} = L^{-3\alpha} (ML^3T^{-2})^\beta M^\gamma,$$

which leads to

$$\begin{cases} -3\alpha + 3\beta & = 0 \\ \beta + \gamma & = 0 \\ -2\beta & = -1 \end{cases}$$

with solution $(\alpha, \beta, \gamma) = (1/2, 1/2, -1/2)$. One thus obtains:

$$\omega_p^2 = \frac{nq_e^2}{m_e \epsilon_0}.$$

Stokes' formula. Let us note that if one writes an expression of the form

$$F \sim \eta^\alpha \rho^\beta v^\gamma R^\delta,$$

dimensional analysis will give only three equations for four unknowns. One thus needs to use the extra assumption given in the text, i.e. the force is proportional to the velocity ($\gamma = 1$). One gets

$$MLT^{-2} = (ML^{-1}T^{-1})^\alpha (ML^{-3})^\beta LT^{-1}L^\delta,$$

from which one deduces the set of equations:

$$\begin{cases} \alpha + \beta & = 1 \\ -\alpha - 3\beta + \delta & = 0 \\ -\alpha & = -1 \end{cases}$$

whose solution reads $(\alpha, \beta, \delta) = (1, 0, 1)$, and thus we have the following scaling

$$F \sim \eta Rv.$$

Stokes' formula actually reads $F = 6\pi\eta Rv$.

♦ **Remark.** The fact that there is not a unique solution if one does not fix the exponent γ comes from the fact that it is possible to build, using the four parameters η , ρ , R , and v , a dimensionless number:

$$\text{Re} \equiv \frac{Rv\rho}{\eta}.$$

This number is called the *Reynolds number* and allows one to determine whether a flow is dominated by viscosity (small Reynolds number $\text{Re} \ll 1$), or by inertia ($\text{Re} \gg 1$). In the latter case, the flow is turbulent. As far as the drag force on a sphere is concerned, it can be written in the most general case $F = \eta Rv f(\text{Re})$, where f is a dimensionless function. Stokes' formula implies that $\lim_{x \rightarrow 0} f(x) = 6\pi$. For large Reynolds numbers, one can expect that the viscosity does not appear any more in the expression of the drag force. By dimensional analysis, one then finds that $F \sim \rho R^2 v^2$: the drag is proportional to the square of the velocity and to the cross-section of the sphere. The function f thus fulfills, when $x \rightarrow \infty$, $f(x) \sim Ax$, where A is a constant.

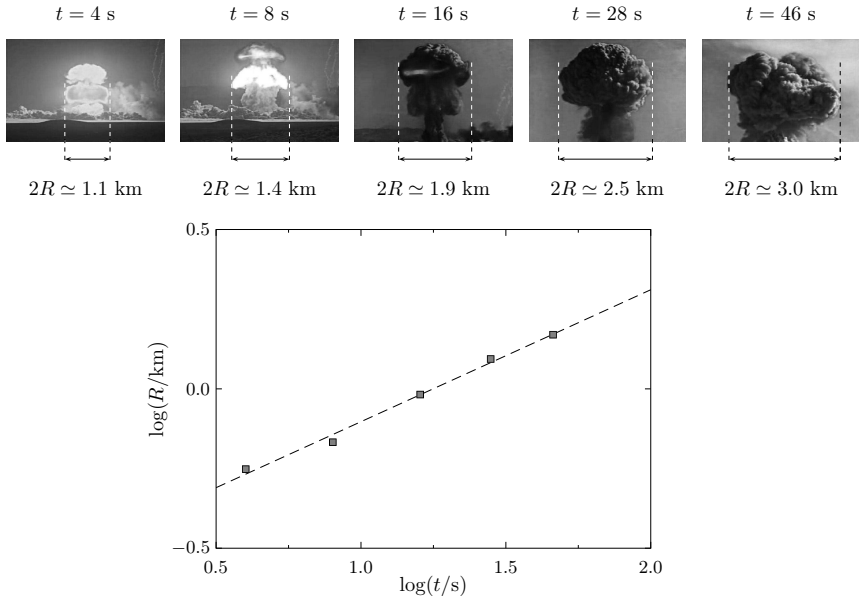


Fig. 1.2 Above, we have indicated for each image the extension $2R(t)$ of the nuclear “mushroom”. Bottom: plot of $\log(R/\text{km})$ versus $\log(t/\text{s})$. One finds a straight line with slope 0.41 (dashed line), very close to the value $2/5 = 0.4$ obtained by dimensional analysis. The intercept is -0.52 .

Energy released in a nuclear explosion. Let us write an equation of the form

$$R \sim \rho^\alpha t^\beta E^\gamma.$$

Dimensional homogeneity implies:

$$L = (ML^{-3})^\alpha T^\beta (ML^2T^{-2})^\gamma,$$

giving the following system:

$$\begin{cases} -3\alpha & + 2\gamma = 1 \\ \alpha & + \gamma = 0 \\ & \beta - 2\gamma = 0 \end{cases}$$

which has the solution $(\alpha, \beta, \gamma) = (-1/5, 2/5, 1/5)$. One thus deduces

$$R \sim \frac{E^{1/5} t^{2/5}}{\rho^{1/5}}.$$

On the series of photographs, one measures the size R of the nuclear “mushroom” (for instance its horizontal diameter $2R$). By plotting $\log(R)$

versus $\log(t)$, one indeed finds a straight line, with a slope close to $2/5$ (see Fig. 1.2). To obtain the energy E released by the explosion, one determines the intercept $\log(E/\rho)/5 = -0.52$ of this line. With $\rho \simeq 1.3 \text{ kg} \cdot \text{m}^{-3}$, one finds $E \sim 3 \times 10^{12} \text{ J}$ (or about one kiloton of TNT, to use the “traditional” unit associated with explosives). The energy released by the bomb dropped onto Hiroshima on August 6th, 1945³ was 15 kilotons of TNT.

³A bomb which caused more than 70,000 casualties...