Chapter 1

Finite-dimensional modules

1.1 The Lie algebra $\mathfrak{sl}_2$ and $\mathfrak{sl}_2$-modules

In what follows we will always work over the field $\mathbb{C}$ of complex numbers. Unless stated otherwise, all vector spaces, tensor products and spaces of homomorphisms are taken over $\mathbb{C}$. As usual, we denote by $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ the sets of integer, rational and real numbers, respectively. We also denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{N}_0$ the set of all non-negative integers.

The Lie algebra $\mathfrak{g} = \mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{C})$ consists of the vector space

$$\mathfrak{sl}_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{C};\ a + d = 0 \right\}$$

of all complex 2×2 matrices with zero trace and the binary bilinear operation $[X, Y] = XY - YX$ of taking the *commutant* of two matrices on this vector space. Here XY denotes the usual (associative) product of the matrices $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ and $Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$ given by the following formula:

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} = \begin{pmatrix} x_{11}y_{11} + x_{12}y_{21} & x_{11}y_{12} + x_{12}y_{22} \\ x_{21}y_{11} + x_{22}y_{21} & x_{21}y_{12} + x_{22}y_{22} \end{pmatrix}.$$

To simplify the notation we will usually denote the Lie algebra $\mathfrak{sl}_2$ simply by $\mathfrak{g}$.

Exercise 1.1. Prove that for any two square complex matrices X and Y of the same size, the matrix $[X, Y]$ has zero trace.

From Exercise 1.1 it follows that the operation $[\cdot, \cdot]$ on $\mathfrak{g}$ is well-defined. The fact that $\mathfrak{g}$ is a *Lie algebra* means that it has the following properties:

Lemma 1.2.

(i) *For any* $X \in \mathfrak{g}$ *we have* $[X, X] = 0$.
(ii) *For any* $X, Y, Z \in \mathfrak{g}$ *we have* $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

Proof. We have $[X, X] = XX - XX = 0$, proving the statement (i). The statement (ii) is proved by the following computation:

$$\begin{aligned}
&[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] \\
&\quad = X(YZ - ZY) - (YZ - ZY)X + Y(ZX - XZ) \\
&\qquad - (ZX - XZ)Y + Z(XY - YX) - (XY - YX)Z \\
&\quad = XYZ - XZY - YZX + ZYX + YZX - YXZ \\
&\qquad - ZXY + XZY + ZXY - ZYX - XYZ + YXZ \\
&\quad = 0.
\end{aligned}$$

□

Exercise 1.3. Show that the condition in Lemma 1.2(i) is equivalent to the following condition: $[X, Y] = -[Y, X]$ for all $X, Y \in \mathfrak{g}$.

The condition in Lemma 1.2(ii) is called the *Jacobi identity*. The assertion of Exercise 1.3 is true over any field of characteristic different from 2 and basically says that the operation $[\cdot, \cdot]$ is *antisymmetric*.

From the definition we have that elements of the algebra $\mathfrak{g}$ are given by four parameters and one non-trivial linear relation. This means that this algebra has dimension three. We now fix the following *natural* or *standard basis* of $\mathfrak{g}$:

$$\mathbf{e} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{h} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

By a direct calculation one gets the following *Cayley table* for the operation $[\cdot, \cdot]$ in the standard basis:

$[\cdot,\cdot]$	$\mathbf{e}$	$\mathbf{f}$	$\mathbf{h}$
$\mathbf{e}$	0	$\mathbf{h}$	$-2\mathbf{e}$
$\mathbf{f}$	$-\mathbf{h}$	0	$2\mathbf{f}$
$\mathbf{h}$	$2\mathbf{e}$	$-2\mathbf{f}$	0

Another way to notate the essential information from the above Cayley table (the diagonal of the table is fairly obvious and given by Lemma 1.2(i)) is the following:

$$\begin{aligned}
[\mathbf{e}, \mathbf{f}] &= \mathbf{ef} - \mathbf{fe} = \mathbf{h}, \\
[\mathbf{h}, \mathbf{e}] &= \mathbf{he} - \mathbf{eh} = 2\mathbf{e}, \\
[\mathbf{h}, \mathbf{f}] &= \mathbf{hf} - \mathbf{fh} = -2\mathbf{f}.
\end{aligned} \tag{1.1}$$

A *module over* $\mathfrak{g}$ (or, simply, a $\mathfrak{g}$-*module*) is a vector space V together with three fixed linear operators $E = E_V$, $F = F_V$ and $H = H_V$ on V, which satisfy the right-hand side equalities in (1.1), that is

$$EF - FE = H, \quad HE - EH = 2E, \quad HF - FH = -2F. \tag{1.2}$$

It is worth noting that the last two relations can be rewritten as follows:

$$HE = E(H + 2), \quad HF = F(H - 2). \tag{1.3}$$

Example 1.4. Let $V = \mathbb{C}$ and $E = F = H = 0$. Then all equalities in (1.2) obviously hold and we get the *trivial* $\mathfrak{g}$-module.

Example 1.5. Let $V = \mathbb{C}^2$. In the usual way we identify the set of all linear operators on V with the set of all complex 2×2 matrices. Set $E = \mathbf{e}$, $F = \mathbf{f}$ and $H = \mathbf{h}$. All equalities in (1.2) hold because of (1.1) and we get the *natural* $\mathfrak{g}$-module.

Example 1.6. Take now $V = \mathfrak{g}$. Let E denote the linear operator on V given by $[\mathbf{e}, \_]$ (that is the linear operator of taking the commutator with $\mathbf{e}$, the latter standing on the left). Analogously define F as $[\mathbf{f}, \_]$ and H as $[\mathbf{h}, \_]$. From Lemma 1.7 below we have that these linear operators satisfy (1.2) and we get the *adjoint* $\mathfrak{g}$-module.

Lemma 1.7. *For any* $X \in \mathfrak{g}$ *we have*

$$[\mathbf{e}, [\mathbf{f}, X]] - [\mathbf{f}, [\mathbf{e}, X]] = [\mathbf{h}, X],$$
$$[\mathbf{h}, [\mathbf{e}, X]] - [\mathbf{e}, [\mathbf{h}, X]] = [2\mathbf{e}, X],$$
$$[\mathbf{h}, [\mathbf{f}, X]] - [\mathbf{f}, [\mathbf{h}, X]] = [-2\mathbf{f}, X].$$

Proof. The equality $[\mathbf{e}, [\mathbf{f}, X]] - [\mathbf{f}, [\mathbf{e}, X]] = [\mathbf{h}, X]$ can be rewritten as follows:

$$[\mathbf{e}, [\mathbf{f}, X]] - [\mathbf{f}, [\mathbf{e}, X]] - [\mathbf{h}, X] = 0. \tag{1.4}$$

Recall that $\mathbf{h} = [\mathbf{e}, \mathbf{f}]$. Applying now Exercise 1.3 to the inner bracket of the second summand and the outer bracket of the third summand reduces the equality (1.4) to the Jacobi identity. Hence the first equality from the formulation follows from Lemma 1.2(ii). The rest is proved similarly. □

Given two $\mathfrak{g}$-modules V and W, a *homomorphism* from V to W (or a $\mathfrak{g}$-*homomorphism*, or, simply, a *morphism*) is a linear map $\Phi : V \to W$ that makes the following diagram commutative for all $X \in \{E, F, H\}$:

$$\begin{array}{ccc} V & \xrightarrow{X_V} & V \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Phi} \\ W & \xrightarrow{X_W} & W \end{array}$$

In other words, the linear map Φ intertwines the actions of $\mathbf{e}$, $\mathbf{f}$ and $\mathbf{h}$ on V and W in the following sense:

$$\Phi E_V = E_W \Phi, \quad \Phi F_V = F_W \Phi, \quad \Phi H_V = H_W \Phi. \tag{1.5}$$

The set of all homomorphisms from V to W is denoted by $\mathrm{Hom}_{\mathfrak{g}}(V, W)$.

Example 1.8. For any two $\mathfrak{g}$-modules V and W, the zero linear map from V to W obviously satisfies (1.5). This is the so-called *zero homomorphism.*

From Example 1.8 it follows that the set $\mathrm{Hom}_{\mathfrak{g}}(V, W)$ is always non-empty.

Exercise 1.9. Show that $\mathrm{Hom}_{\mathfrak{g}}(V, W)$ is closed with respect to the usual addition of linear maps and multiplication of linear maps by complex numbers. In particular, show that the set $\mathrm{Hom}_{\mathfrak{g}}(V, W)$ is a vector space.

Example 1.10. For any $\mathfrak{g}$-module V the identity map id_V on V obviously satisfies (1.5) (where $V = W$). This is the so-called *identity homomorphism.*

An injective homomorphism is called a *monomorphism*, a surjective homomorphism is called an *epimorphism* and a bijective homomorphism is called an *isomorphism.* As usual, it only makes sense to study $\mathfrak{g}$-modules up to isomorphism. The fact that two modules V and W are isomorphic is usually denoted by $V \cong W$.

Let V be a $\mathfrak{g}$-module. A subspace $W \subset V$ is called an *submodule* (or a $\mathfrak{g}$-*submodule*) of V provided that it is invariant with respect to the action of E_V, F_V and H_V, that is

$$E_V W \subset W, \quad F_V W \subset W, \quad H_V W \subset W. \tag{1.6}$$

For example, the module V always has two obvious submodules; namely, the zero subspace and the whole space V. Any submodule, different from these two is called a *proper* submodule. A module which does not have any proper submodules is called *simple*. For example, any module of dimension one is simple.

Exercise 1.11. Show that all $\mathfrak{g}$-modules from Examples 1.4, 1.5 and 1.6 are simple.

Exercise 1.12. Let $V = \mathbb{C}^2$ and $E = F = H = 0$. Show that this defines on V the structure of a $\mathfrak{g}$-module that is not simple.

Exercise 1.13. Let V be a $\mathfrak{g}$-module and W a submodule of V. Show that the quotient space V/W carries the natural structure of a $\mathfrak{g}$-module given by $E(v+W) = E(v)+W$, $F(v+W) = F(v)+W$ and $H(v+W) = H(v)+W$. The module V/W is called the *quotient* or the *factor* of V by W.

Lemma 1.14. *Let V and W be two $\mathfrak{g}$-modules and $\Phi \in \mathrm{Hom}_{\mathfrak{g}}(V, W)$. Then*

(i) The kernel $\mathrm{Ker}(\Phi)$ of Φ is a submodule of V.
(ii) The image $\mathrm{Im}(\Phi)$ of Φ is a submodule of W.

Proof. Let $v \in \mathrm{Ker}(\Phi)$ and $X \in \{E, F, H\}$. This gives us

$$\Phi(X_V(v)) = \Phi X_V(v) \stackrel{(1.5)}{=} X_W \Phi(v) = 0,$$

implying $X_V(v) \in \mathrm{Ker}(\Phi)$. This proves (i). To prove (ii) is left as an exercise to the reader. □

1.2 Classification of simple finite-dimensional modules

This section contains perhaps the most classical part of the $\mathfrak{sl}_2$-representation theory; namely, a classification of all simple *finite-dimensional* $\mathfrak{sl}_2$-modules. As we will see later, such modules form only a very small family of simple $\mathfrak{sl}_2$-modules. A description of *all* simple $\mathfrak{sl}_2$-modules is an ultimate goal of this book, but that will require much more theory and effort. The beauty of finite-dimensional modules is in the fact that their description is absolutely elementary.

Let $V \neq 0$ be a finite-dimensional $\mathfrak{g}$-module. For $\lambda \in \mathbb{C}$ set

$$V(\lambda) = \{v \in V \,:\, (H-\lambda)^k v = 0 \text{ for some } k \in \mathbb{N}\},$$
$$V_\lambda = \{v \in V \,:\, Hv = \lambda v\}$$

(here, as usual, we identify $\mathbb{C}$ with multiples of id_V).

As we are working over the algebraically closed field of complex numbers, from the Jordan Decomposition Theorem we have that

$$V \cong \bigoplus_{\lambda\in\mathbb{C}} V(\lambda). \tag{1.7}$$

Set $W = \bigoplus_{\lambda\in\mathbb{C}} V_\lambda \subset V$ and note that $W \neq 0$ as H must have at least one eigenvalue and hence at least one non-zero eigenvector in V.

Lemma 1.15. *Let $\lambda \in \mathbb{C}$.*

(i) $EV(\lambda) \subset V(\lambda+2)$ and $EV_\lambda \subset V_{\lambda+2}$.
(ii) $FV(\lambda) \subset V(\lambda-2)$ and $FV_\lambda \subset V_{\lambda-2}$.
(iii) $HV(\lambda) \subset V(\lambda)$ and $HV_\lambda \subset V_\lambda$.

Proof. For $v \in V_\lambda$ we have

$$H(E(v)) = HEv \overset{(1.2)}{=} EHv + 2Ev = \lambda Ev + 2Ev = (\lambda+2)E(v),$$

which implies the second part of the statement (i). Similarly, for $v \in V(\lambda)$ let $k \in \mathbb{N}_0$ be such that $(H-\lambda)^k v = 0$. Then we have

$$\begin{aligned}(H-(\lambda+2))^k(E(v)) &= (H-(\lambda+2))^k Ev \overset{(1.3)}{=} \\ &= E(H+2-(\lambda+2))^k v = E(H-\lambda)^k v = 0,\end{aligned}$$

which implies the first part of the statement (i).

The statement (ii) is proved similarly. The statement (iii) is obvious from the definitions. □

Exercise 1.16. Generalizing the arguments from the proof of Lemma 1.15, show that for any $f(x) \in \mathbb{C}[x]$ one has the equalities $f(H)E = Ef(H+2)$ and $f(H)F = Ff(H-2)$.

From Lemma 1.15 we immediately obtain:

Corollary 1.17. *The space W is a submodule of V, in particular, we have $W = V$ if the module V is simple.*

If the module V is simple, we can use Corollary 1.17 to improve the decomposition given by (1.7) as follows:

$$V \cong \bigoplus_{\lambda\in\mathbb{C}} V_\lambda. \tag{1.8}$$

Since V is finite-dimensional, the decomposition (1.8) must be finite in the sense that only finitely many summands are non-zero. Thus we can fix some $\mu \in \mathbb{C}$ such that $V_\mu \neq 0$ and $V_{\mu+2k} = 0$ for all $k \in \mathbb{N}$. Let $v \in V_\mu$ be some non-zero element. As $V_{\mu-2k}$ must be zero for some $k \in \mathbb{N}$, from Lemma 1.15(ii) it follows that there exists a minimal $n \in \mathbb{N}$ such that $F^n v = 0$. For $i \in \{1, 2, \ldots, n-1\}$ set $v_i = F^i v$, and also set $v_0 = v$. From (1.8) it follows that the v_i's are linearly independent. From Lemma 1.15(ii) we have $Hv_i = (\mu - 2i)v_i$ for all i. Let N denote the linear span of all v_i's.

Lemma 1.18. *We have* $Ev_0 = 0$ *and* $Ev_i = i(\mu - i + 1)v_{i-1}$ *for all* $i \in \{1, 2, \ldots, n-1\}$.

Proof. That $Ev_0 = 0$ is obvious. To prove the rest we proceed by induction on i. For $i = 1$ we have

$$Ev_1 = EFv_0 \overset{(1.2)}{=} FEv_0 + Hv_0 = 0 + \mu v_0 = \mu v_0,$$

which agrees with our formula. When $i > 1$ for the induction step we have:

$$\begin{aligned} Ev_i &= EFv_{i-1} \\ &\overset{(1.2)}{=} FEv_{i-1} + Hv_{i-1} \\ \text{(inductive assumption)} \quad &= (i-1)(\mu - i + 2)Fv_{i-2} + (\mu - 2(i-1))v_{i-1} \\ &= i(\mu - i + 1)v_{i-1}. \end{aligned}$$

This completes the proof. □

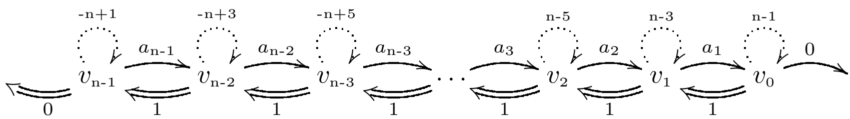
Corollary 1.19. *N is a submodule of V, in particular, $N = V$ provided that V is simple.*

Proof. That N is invariant with respect to the action of H and F is obvious. By Lemma 1.18 it is also invariant with respect to the action of E. The claim follows. □

Lemma 1.20. $\mu = n - 1$.

Proof. From the inductive argument used in the proof of Lemma 1.18 we get $EFv_{n-1} = n(\mu - n + 1)v_{n-1}$. However, $Fv_{n-1} = 0$ by our assumptions, hence $n(\mu - n + 1) = 0$ implying $\mu = n - 1$. □

Assuming that V is simple, let us sum up the information which we now have about this module. It has the basis $\{v_0, v_1, \ldots, v_{n-1}\}$, in which the action of the operators E, F and H can be depicted as follows:



(1.9)

Here $a_i = i(n-i)$. The double arrow represents the action of F, the regular arrow represents the action of E and the dotted arrow represents the action of H. The numbers over arrows are coefficients.

Exercise 1.21. Check that for any $n \in \mathbb{N}$ the picture (1.9) defines on the formal linear span of $\{v_0, \dots, v_{n-1}\}$ the structure of a $\mathfrak{g}$-module. We will denote this module by $\mathbf{V}^{(n)}$.

Now we are ready to formulate the main result of this section.

Theorem 1.22 (Classification of simple finite-dimensional modules).

(i) For every $n \in \mathbb{N}$ *the module* $\mathbf{V}^{(n)}$ *is a simple* $\mathfrak{g}$*-module of dimension* n.

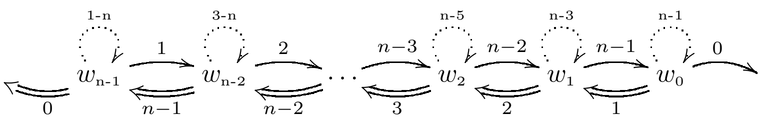
(ii) For any $n, m \in \mathbb{N}$ *we have* $\mathbf{V}^{(n)} \cong \mathbf{V}^{(m)}$ *if and only if* $n = m$.

(iii) Let V *be a simple finite-dimensional* $\mathfrak{g}$*-module of dimension* n*. Then* $V \cong \mathbf{V}^{(n)}$.

Proof. That $\mathbf{V}^{(n)}$ is a module follows from Exercise 1.21. Let $M \subset \mathbf{V}^{(n)}$ be a non-zero submodule and $v \in M$, $v \neq 0$. From (1.9) we have that $E^n v = 0$, in particular, $E^n M = 0$ and hence M must have a non-trivial intersection with the kernel of E. Again, from (1.9) it follows that the kernel of E is just the linear span of v_0 and is, in particular, one-dimensional. Hence M contains v_0. Applying to v_0 the operator F inductively we get that M contains all the v_i's. Hence $M = \mathbf{V}^{(n)}$. This proves the statement (i).

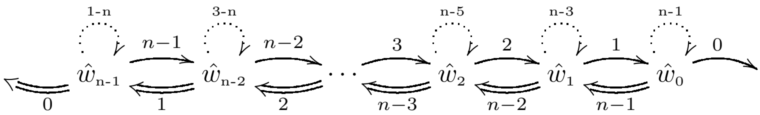
As $\dim \mathbf{V}^{(n)} = n$, the statement (ii) is obvious. The statement (iii) follows from the analysis leading to the picture (1.9). □

Exercise 1.23. Show that after rescaling the basis $\{v_i\}$ in the following way: $w_i = \frac{1}{i!} v_i$ the picture (1.9) transforms into the following symmetric form:



(1.10)

Exercise 1.24. Show that one can rescale the basis $\{v_i\}$ so that in the new basis $\{\hat{w}_i\}$ the picture (1.9) transforms into the following symmetric form:



(1.11)

Exercise 1.25. Let V be a simple finite-dimensional $\mathfrak{g}$-module which contains a non-zero vector v such that $E(v) = 0$ and $H(v) = (n-1)v$. Show that $V \cong \mathbf{V}^{(n)}$.

In the basis $\{w_0, w_1, \dots, w_{n-1}\}$ from Exercise 1.23 the linear operators E, F and H are given by the following matrices:

$$E = \begin{pmatrix} 0 & n\text{-}1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & n\text{-}2 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 2 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 2 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & n\text{-}2 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & n\text{-}1 & 0 \end{pmatrix}$$

$$H = \begin{pmatrix} n\text{-}1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & n\text{-}3 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & n\text{-}5 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 5\text{-}n & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 3\text{-}n & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1\text{-}n \end{pmatrix}$$

We complete this section with a description of homomorphisms between simple modules:

Theorem 1.26 (Schur's lemma).

(i) Any non-zero homomorphism between two simple $\mathfrak{g}$-modules is an isomorphism.

(ii) For any two simple finite-dimensional $\mathfrak{g}$-modules V and W we have

$$\mathrm{Hom}_{\mathfrak{g}}(V, W) \cong \begin{cases} \mathbb{C}, & V \cong W; \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. Let $\Phi \in \mathrm{Hom}_{\mathfrak{g}}(V, W)$ be some non-zero homomorphism. Applying Lemma 1.14(i) we have that the kernel of Φ is a submodule of V. As V is simple and $\Phi \neq 0$, we thus get that $\mathrm{Ker}(\Phi) = 0$ and Φ is injective. By Lemma 1.14(ii) the image of Φ is a submodule of W. As W is simple and $\Phi \neq 0$, we thus get that $\mathrm{Im}(\Phi) = W$ and Φ is surjective. Therefore any non-zero element of $\mathrm{Hom}_{\mathfrak{g}}(V, W)$ is an isomorphism, proving the statement (i). In particular, $\mathrm{Hom}_{\mathfrak{g}}(V, W) = 0$ if $V \ncong W$.

Assume now that $V \cong W$ and $0 \neq \Psi \in \mathrm{Hom}_{\mathfrak{g}}(V, W)$. We then have an obvious isomorphism from $\mathrm{Hom}_{\mathfrak{g}}(V, V)$ to $\mathrm{Hom}_{\mathfrak{g}}(V, W)$ given by $\Phi \mapsto \Psi \circ \Phi$. Let us show that

$$\mathrm{Hom}_{\mathfrak{g}}(V, V) = \mathbb{C}\,\mathrm{id}_V \cong \mathbb{C}.$$

If $\Phi \in \mathrm{Hom}_{\mathfrak{g}}(V, V)$ is non-zero, it must have a non-zero eigenvalue, $\lambda \in \mathbb{C}$ say (as $\mathbb{C}$ is algebraically closed). Then $\Phi - \lambda \cdot \mathrm{id}_V \in \mathrm{Hom}_{\mathfrak{g}}(V, V)$. However, any eigenvector of Φ with eigenvalue λ belongs to the kernel of $\Phi - \lambda \cdot \mathrm{id}_V$. Thus $\Phi - \lambda \cdot \mathrm{id}_V$ is not an isomorphism, and hence it must be zero by the previous paragraph. This yields $\Phi = \lambda \cdot \mathrm{id}_V$ and completes the proof. □

1.3 Semi-simplicity of finite-dimensional modules

Given two $\mathfrak{g}$-modules V and W the vector space $V \oplus W$ can be endowed with the structure of a $\mathfrak{g}$-module as follows:

$$\begin{aligned} E(v \oplus w) &= E(v) \oplus E(w), \\ F(v \oplus w) &= F(v) \oplus F(w), \\ H(v \oplus w) &= H(v) \oplus H(w). \end{aligned} \tag{1.12}$$

The module $V \oplus W$ is called the *direct sum* of V and W. For $n \in \mathbb{N}$ and any $\mathfrak{g}$-module V we denote by nV the $\mathfrak{g}$-module

$$\underbrace{V \oplus V \oplus \cdots \oplus V}_{n \text{ summands}}.$$

Exercise 1.27. Check that the formulae (1.12) indeed define on $V \oplus W$ the structure of a $\mathfrak{g}$-module.

A $\mathfrak{g}$-module V is called *decomposable* provided that $V \cong V_1 \oplus V_2$ for some non-zero $\mathfrak{g}$-modules V_1 and V_2. Those $\mathfrak{g}$-modules which are not decomposable are called *indecomposable*. A module, which is isomorphic to a direct sum of (possibly finitely many) simple modules is called *semi-simple*.

Example 1.28. Every simple module is indecomposable; in particular, every one-dimensional module is indecomposable. Indeed, if $V \cong V_1 \oplus V_2$ and both V_1 and V_2 are non-zero, then V_1 is a proper submodule of V and hence V cannot be simple.

In the general situation (for example if one considers all $\mathfrak{g}$-modules) there exist many indecomposable modules which are not simple. We will see many examples later on. However, the case of finite-dimensional modules turns out to be very special. The main aim of the present section is to prove the following statement:

Theorem 1.29 (Weyl's Theorem). *Every indecomposable finite-dimensional $\mathfrak{g}$-module is simple. Equivalently, every finite-dimensional $\mathfrak{g}$-module is semi-simple.*

To prove this theorem we will need some preparation. From now and until the end of the proof we assume that V is a finite-dimensional $\mathfrak{g}$-module. Consider the *Casimir operator* $C = C_V$ on V, defined as follows:

$$C = (H+1)^2 + 4FE.$$

Lemma 1.30.

(i) $C = (H-1)^2 + 4EF = H^2 + 1 + 2EF + 2FE$.
(ii) $HC = CH$, $EC = CE$, $FC = CF$.

Proof. The statement (i) follows from the definition of C and the equality $EF = FE + H$. To prove the statement (ii) we use (i) and Exercise 1.16 as follows:

$$\begin{aligned} HC &= H((H+1)^2 + 4FE) \\ &= H(H+1)^2 + 4HFE \\ \text{(by Exercise 1.16)} &= (H+1)^2 H + 4F(H-2)E \\ \text{(by Exercise 1.16)} &= (H+1)^2 H + 4FEH \\ &= ((H+1)^2 + 4FE)H \\ &= CH; \end{aligned}$$

$$\begin{aligned} EC &= E((H+1)^2 + 4FE) \\ &= E(H+1)^2 + 4EFE \\ \text{(by Exercise 1.16)} &= (H-1)^2E + 4EFE \\ &= ((H-1)^2 + 4EF)E \\ \text{by (i)} &= CE. \end{aligned}$$

The equality $FC = CF$ is checked in the same way. □

To proceed, we need to recall the following result from linear algebra:

Exercise 1.31. Let W be a vector space, A and B two linear commuting operators on the space W and $\lambda \in \mathbb{C}$. Show that both the subspace

$$\{w \in W \,:\, Aw = \lambda w\}$$

and the subspace

$$\{w \in W \,:\, (A-\lambda)^k w = 0 \text{ for some } k \in \mathbb{N}\}$$

are invariant with respect to B.

Applying the Jordan Decomposition Theorem to the linear operator C on V we find that

$$V \cong \bigoplus_{\tau\in\mathbb{C}} V(C,\tau),$$

where

$$V(C,\tau) = \{v \in V \,:\, (C-\tau)^k v = 0 \text{ for some } k \in \mathbb{N}\}.$$

Lemma 1.32. *For any $\tau \in \mathbb{C}$ the subspace $V(C,\tau)$ is a $\mathfrak{g}$-submodule of V. In particular, if V is indecomposable, then $V = V(C,\tau)$ for some $\tau \in \mathbb{C}$.*

Proof. By Lemma 1.30(ii), the operator C commutes with the operators E, F and H. Hence all these operators preserve $V(C,\tau)$ by Exercise 1.31. The claim follows. □

Exercise 1.33. Check that $C_{\mathbf{V}^{(n)}} = n^2 \cdot \mathrm{id}_{\mathbf{V}^{(n)}}$ for all $n \in \mathbb{N}$.

Now we are ready to prove Theorem 1.29:

Proof. Let V be a non-zero indecomposable finite-dimensional $\mathfrak{g}$-module. This means that it has a non-trivial simple submodule and hence from Lemma 1.32 and Exercise 1.33 we obtain that $V = V(C, n^2)$ for some $n \in \mathbb{N}$.

Consider the decomposition (1.7). First, we claim that E acts injectively on any $V(\lambda)$, $\lambda \neq n-1, -n-1$. Indeed, for any $v \in V(\lambda) \cap \mathrm{Ker}(E)$ we have

$$E(H(v)) = EHv \overset{(1.2)}{=} HEv - 2Ev = 0,$$

and hence $V(\lambda) \cap \mathrm{Ker}(E)$ is invariant under the action of H. If we have the inequality $V(\lambda) \cap \mathrm{Ker}(E) \neq 0$, then $V_\lambda \cap \mathrm{Ker}(E) \neq 0$ and for any $v \in V_\lambda \cap \mathrm{Ker}(E)$ we have

$$Cv = ((H+1)^2 + 4FE)v = (H+1)^2 v + 4FEv = (\lambda+1)^2 v.$$

At the same time $Cv = n^2 v$ as $V = V(C, n^2)$, which implies $\lambda = n-1$ or $\lambda = -n-1$. Analogously, one can show that F acts injectively on any $V(\lambda)$ such that $\lambda \neq 1-n, n+1$.

Since V is finite-dimensional, the previous paragraph implies that the inequality $V(\lambda) \neq 0$ is possible only if $\lambda \in \{-n+1, -n+3, \dots, n-1\}$ and that $\mathrm{Ker}(E) = V(n-1)$, $\mathrm{Ker}(F) = V(1-n)$. In particular, $\dim V(\lambda) = \dim V(\mu)$ for any $\lambda, \mu \in \{-n+1, -n+3, \dots, n-1\}$. Furthermore, for any $i \in \{1, 2, \dots, n-1\}$ the restriction A_i of the operator F^i to the subspace $V(n-1)$ gives a linear isomorphism from $V(n-1)$ to $V(n-1-2i)$. Hence we can identify $V(n-1)$ and $V(n-1-2i)$ as vector spaces via the action of A_i. Set $A = A_{n-1}$.

As C commutes with H, all $V(\lambda)$'s are invariant with respect to C. Denote by C_1 and H_1 the restrictions of C and H to $V(n-1)$, respectively. Denote by C_2 and H_2 the restrictions of C and H to $V(1-n)$ respectively. Restricting $CF^{n-1} = F^{n-1}C$ to $V(n-1)$ we get

$$AC_1 = C_2 A. \tag{1.13}$$

Analogously, using $FH = (H+2)F$ (see Exercise 1.16) we get

$$AH_1 = (H_2 + 2(n-1))A. \tag{1.14}$$

As $\mathrm{Ker}(E) = V(n-1)$ and $C = (H+1)^2 + 4FE$, we have

$$C_1 = (H_1 + 1)^2. \tag{1.15}$$

As $\mathrm{Ker}(F) = V(1-n)$ and $C = (H-1)^2 + 4EF$, we have

$$C_2 = (H_2 - 1)^2. \tag{1.16}$$

Thus we have:

$$\begin{aligned}(H_1+1)^2 &\overset{(1.15)}{=} C_1\\ &= A^{-1}AC_1\\ \text{(by (1.13))} &= A^{-1}C_2A\\ \text{(by (1.16))} &= A^{-1}(H_2-1)^2A\\ \text{(by (1.14))} &= A^{-1}A(H_1-1-2(n-1))^2\\ &= (H_1-1-2(n-1))^2.\end{aligned}$$

Hence $(H_1+1)^2 = (H_1-1-2(n-1))^2$, which reduces to $H_1 = n-1$. This means that $V(n-1) = V_{n-1}$. Since $A_iH = (H+2i)A_i$ and A_i identifies the space $V(n-1)$ with the space $V(n-1-2i)$ for all i, we get $V(\lambda) = V_\lambda$ for all $\lambda \in \{-n+1, -n+3, \dots, n-1\}$.

Let $\{v_1, \dots, v_k\}$ be a basis of V_{n-1}. For $i \in \{1, \dots, k\}$ denote by W_i the linear span of $\{v_i, FV_i, \dots, F^{n-1}v_i\}$. From the above we have

$$V \cong W_1 \oplus W_2 \oplus \cdots \oplus W_k$$

and, by Corollary 1.19, each W_i is a submodule of V. Since V is indecomposable by our assumptions, we get $k = 1$ and thus $\dim V_{n-1} = 1$. In this case Corollary 1.19 and (1.9) imply that $V \cong \mathbf{V}^{(n)}$, which completes the proof. □

Corollary 1.34. *Let V be a finite-dimensional $\mathfrak{g}$-module. Then*

$$V \cong \bigoplus_{n\in\mathbb{N}} m_n \mathbf{V}^{(n)},$$

where

$$m_n = \dim \mathrm{Hom}_{\mathfrak{g}}(\mathbf{V}^{(n)}, V) = \dim \mathrm{Hom}_{\mathfrak{g}}(V, \mathbf{V}^{(n)}).$$

Proof. From Theorem 1.29 it follows that we can decompose V into a direct sum of simple modules, say $V \cong V_1 \oplus \cdots \oplus V_k$, where all the V_i's are simple. Now

$$\begin{aligned}\dim \mathrm{Hom}_{\mathfrak{g}}(\mathbf{V}^{(n)}, V) &= \dim \mathrm{Hom}_{\mathfrak{g}}(\mathbf{V}^{(n)}, \oplus_i V_i)\\ &= \sum_i \dim \mathrm{Hom}_{\mathfrak{g}}(\mathbf{V}^{(n)}, V_i)\\ \text{(by Schur's lemma)} &= |\{i : \mathbf{V}^{(n)} \cong V_i\}|.\end{aligned}$$

This proves the first equality for m_n and the second equality is proved similarly. □

1.4 Tensor products of finite-dimensional modules

Given two $\mathfrak{g}$-modules V and W the vector space $V \otimes W$ can be endowed with the structure of a $\mathfrak{g}$-module as follows:

$$\begin{aligned} E(v \otimes w) &= E(v) \otimes w + v \otimes E(w), \\ F(v \otimes w) &= F(v) \otimes w + v \otimes F(w), \\ H(v \otimes w) &= H(v) \otimes w + v \otimes H(w). \end{aligned} \tag{1.17}$$

The module $V \otimes W$ is called the *tensor product* of V and W. For $n \in \mathbb{N}$ and any $\mathfrak{g}$-module V we denote by $V^{\otimes n}$ the $\mathfrak{g}$-module

$$\underbrace{V \otimes V \otimes \cdots \otimes V}_{n \text{ factors}}.$$

Exercise 1.35. Check that the formulae (1.17) do indeed define on $V \otimes W$ the structure of a $\mathfrak{g}$-module.

Exercise 1.36. Let V and W be two $\mathfrak{g}$-modules. Check that the map $v \otimes w \mapsto w \otimes v$ induces an isomorphism between $V \otimes W$ and $W \otimes V$.

Exercise 1.37. Let V_1, V_2 and W be $\mathfrak{g}$-modules. Prove that

$$(V_1 \oplus V_2) \otimes W \cong (V_1 \otimes W) \oplus (V_2 \otimes W).$$

Exercise 1.38. Let V, W and U be $\mathfrak{g}$-modules. Prove that

$$V \otimes (W \otimes U) \cong (V \otimes W) \otimes U.$$

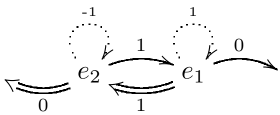
If both V and W are finite-dimensional, the module $V \otimes W$ is finite-dimensional as well. Due to Corollary 1.34, it is natural to ask how $V \otimes W$ decomposes into a direct sum of simple modules (depending on V and W). Exercises 1.36 and 1.37 mean that to answer this question it is sufficient to consider the case when both V and W are simple modules. This is what we will do in this section. Our main result is the following:

Theorem 1.39. *Let $m, n \in \mathbb{N}$ be such that $m \leq n$. Then*

$$\mathbf{V}^{(n)} \otimes \mathbf{V}^{(m)} \cong \mathbf{V}^{(n-m+1)} \oplus \mathbf{V}^{(n-m+3)} \oplus \cdots \oplus \mathbf{V}^{(n+m-3)} \oplus \mathbf{V}^{(n+m-1)}. \tag{1.18}$$

Proof. We proceed by induction on m. If $m = 1$, the module $\mathbf{V}^{(1)} \cong \mathbb{C}$ is the trivial $\mathfrak{g}$-module and hence $\mathbf{V}^{(n)} \otimes \mathbf{V}^{(1)}$ is isomorphic to $\mathbf{V}^{(n)}$, for example via the isomorphism $v \otimes 1 \mapsto v$.

Let $m = 2$. Then $\mathbf{V}^{(2)}$ is the natural $\mathfrak{g}$-module. Let $\{e_1, e_2\}$ be the natural basis of $\mathbf{V}^{(2)}$. Then the action of E, F and H in this basis is given by the following picture (see (1.9)):



(1.19)

Assume that $\mathbf{V}^{(n)}$ is given by (1.9). Then from the formulae (1.17) we obtain that the vector $v_0 \otimes e_1 \in \mathbf{V}^{(n)} \otimes \mathbf{V}^{(2)}$ satisfies $E(v_0 \otimes e_1) = 0$ and $H(v_0 \otimes e_1) = n(v_0 \otimes e_1)$. The only $\mathbf{V}^{(i)}$ which contains a non-zero vector with such properties is $\mathbf{V}^{(n+1)}$ (see Exercise 1.25). Hence $\mathbf{V}^{(n+1)}$ is a direct summand of $\mathbf{V}^{(n)} \otimes \mathbf{V}^{(2)}$.

Let $w = v_1 \otimes e_1 - (n-1)v_0 \otimes e_2 \neq 0$. Using the definitions, one can easily check that $E(w) = 0$ and $H(w) = (n-2)w$. The only $\mathbf{V}^{(i)}$ that contains a non-zero vector with such properties is $\mathbf{V}^{(n-1)}$ (Exercise 1.25). Hence $\mathbf{V}^{(n-1)}$ is a direct summand of $\mathbf{V}^{(n)} \otimes \mathbf{V}^{(2)}$. But

$$\begin{aligned}\dim \mathbf{V}^{(n)} \otimes \mathbf{V}^{(2)} &= \dim \mathbf{V}^{(n)} \times \dim \mathbf{V}^{(2)}\\ &= 2n\\ &= (n-1) + (n+1)\\ &= \dim \mathbf{V}^{(n-1)} + \dim \mathbf{V}^{(n+1)}.\end{aligned}$$

This implies that

$$\mathbf{V}^{(n)} \otimes \mathbf{V}^{(2)} \cong \mathbf{V}^{(n-1)} \oplus \mathbf{V}^{(n+1)}. \tag{1.20}$$

Now let us prove the induction step. We assume that $k > 2$ and that (1.18) is true for all $m = 1, \dots, k-1$. Let us compute $\mathbf{V}^{(n)} \otimes \mathbf{V}^{(k-1)} \otimes \mathbf{V}^{(2)}$ in two different ways. On the one hand we have

$$\begin{aligned}\mathbf{V}^{(n)} \otimes \mathbf{V}^{(k-1)} \otimes \mathbf{V}^{(2)} &\overset{(1.20)}{\cong} \mathbf{V}^{(n)} \otimes (\mathbf{V}^{(k)} \oplus \mathbf{V}^{(k-2)})\\ \text{(by Exercise 1.37)} &\cong \mathbf{V}^{(n)} \otimes \mathbf{V}^{(k)} \oplus \mathbf{V}^{(n)} \otimes \mathbf{V}^{(k-2)}\\ \text{(by inductive assumption)} &\cong \mathbf{V}^{(n)} \otimes \mathbf{V}^{(k)} \oplus \mathbf{V}^{(n-k+3)} \oplus \dots\\ &\quad \dots \oplus \mathbf{V}^{(n+k-5)} \oplus \mathbf{V}^{(n+k-3)}.\end{aligned} \tag{1.21}$$

On the other hand, using the inductive assumption we have

$$\begin{aligned}
\mathbf{V}^{(n)} \otimes \mathbf{V}^{(k-1)} \otimes \mathbf{V}^{(2)} &\cong \left(\bigoplus_{i=0}^{k-2} \mathbf{V}^{(n-k+2+2i)}\right) \otimes \mathbf{V}^{(2)} \\
\text{(by Exercise 1.37)} &\cong \bigoplus_{i=0}^{k-2} \mathbf{V}^{(n-k+2+2i)} \otimes \mathbf{V}^{(2)} \\
\text{(by (1.20))} &\cong \bigoplus_{i=0}^{k-2} \left(\mathbf{V}^{(n-k+3+2i)} \oplus \mathbf{V}^{(n-k+1+2i)}\right) \\
&\cong \mathbf{V}^{(n-k+1)} \oplus \mathbf{V}^{(n-k+3)} \oplus \dots \oplus \mathbf{V}^{(n+k-1)} \oplus \\
&\quad \oplus \mathbf{V}^{(n-k+3)} \oplus \mathbf{V}^{(n-k+5)} \oplus \dots \oplus \mathbf{V}^{(n+k-3)}.
\end{aligned} \tag{1.22}$$

The statement of the theorem now follows, comparing (1.21) with (1.22) and using the uniqueness of the decomposition of $\mathbf{V}^{(n)} \otimes \mathbf{V}^{(k-1)} \otimes \mathbf{V}^{(2)}$ into a direct sum of irreducible modules (Corollary 1.34). □

1.5 Unitarizability of finite-dimensional modules

The correspondence

$$\mathbf{e}^\star = \mathbf{f}, \quad \mathbf{f}^\star = \mathbf{e}, \quad \mathbf{h}^\star = \mathbf{h}$$

uniquely extends to a *skew-linear* involution $\star$ on the vector space $\mathfrak{g}$ in the sense that $(\lambda x)^\star = \overline{\lambda} x^\star$ for all $x \in \mathfrak{g}$ and $\lambda \in \mathbb{C}$, where $\bar{\cdot}$ denotes the complex conjugation. This involution satisfies

$$[x^\star, y^\star] = [y, x]^\star$$

for all $x, y \in \mathfrak{g}$ and hence is a (skew) *anti-involution* of the Lie algebra $\mathfrak{g}$. The involution $\star$ induces an involution on the set $\{E, F, H\}$, which we will denote by the same symbol.

A $\mathfrak{g}$-module V is called *unitarizable* with respect to the involution $\star$ provided that there exists a (positive definite) Hermitian inner product $(\cdot, \cdot)$ on V such that

$$(X(v), w) = (v, X^\star(w)) \tag{1.23}$$

for all $v, w \in V$ and $X \in \{E, F, H\}$. The aim of this section is to prove the following result:

Theorem 1.40. *Every finite-dimensional $\mathfrak{g}$-module is unitarizable.*

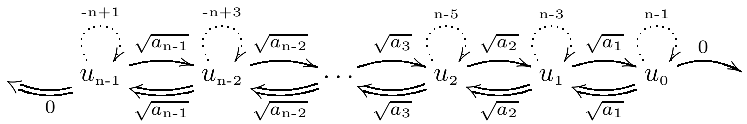
Exercise 1.41. Show that a direct sum $V \oplus W$ of two $\mathfrak{g}$-modules V and W is unitarizable if, and only if, each summand is unitarizable.

Proof. By Corollary 1.34, every finite-dimensional $\mathfrak{g}$-module decomposes into a direct sum of modules $\mathbf{V}^{(n)}$, $n \in \mathbb{N}$. Hence Exercise 1.41 implies that it is enough to prove the statement of the theorem for the modules $\mathbf{V}^{(n)}$, $n \in \mathbb{N}$.

Assume that $n \in \mathbb{N}$ and the module $\mathbf{V}^{(n)}$ is given by (1.9). Note that all $a_i > 0$ and define

$$c_0 = 1, \qquad c_i = \frac{c_{i-1}}{\sqrt{a_i}},\ i = 1, \dots, n-1.$$

Then $u_i = c_i v_i$ defines a diagonal change of basis in $\mathbf{V}^{(n)}$. In the basis $\{u_i\}$ the action of E, F and H is given by:



(1.24)

Let $(\cdot,\cdot)$ be the inner product on $\mathbf{V}^{(n)}$ with respect to which the basis $\{u_0, \dots, u_{n-1}\}$ is orthonormal. From (1.24) it follows by a direct calculation that in this basis the linear operators E, F and H satisfy (1.23). This proves that $\mathbf{V}^{(n)}$ is unitarizable. As mentioned above, the general statement follows. $\square$

The anti-involution $\star$ is not the only anti-involution on $\mathfrak{g}$. The correspondence

$$\mathbf{e}^{\diamond} = \mathbf{e}, \quad \mathbf{f}^{\diamond} = \mathbf{f}, \quad \mathbf{h}^{\diamond} = -\mathbf{h}$$

uniquely extends to a linear involution $\diamond$ on $\mathfrak{g}$. This involution satisfies

$$[x^{\diamond}, y^{\diamond}] = [y, x]^{\diamond}$$

for all $x, y \in \mathfrak{g}$ and hence is an anti-involution of the Lie algebra $\mathfrak{g}$. The involution $\diamond$ induces an involution on the set $\{E, F, \pm H\}$, which we will denote by the same symbol. A $\mathfrak{g}$-module V is called a $\diamond$-*module* provided that there exists a non-degenerate symmetric bilinear form $(\cdot,\cdot)$ on V such that

$$(X(v), w) = (v, X^{\diamond}(w)) \tag{1.25}$$

for all $v, w \in V$ and $X \in \{E, F, H\}$.

Exercise 1.42. Let V be a non-trivial simple finite-dimensional $\diamond$-module with the corresponding symmetric bilinear form $(\cdot,\cdot)$. Show that $(\cdot,\cdot)$ is neither positive nor negative definite.

Exercise 1.43. Show that a direct sum $V \oplus W$ of two $\diamond$-modules is a $\diamond$-module.

Theorem 1.44. *Every finite-dimensional* $\mathfrak{g}$*-module is a* $\diamond$*-module.*

Proof. By Corollary 1.34, every finite-dimensional $\mathfrak{g}$-module decomposes into a direct sum of modules $\mathbf{V}^{(n)}$, $n \in \mathbb{N}$. Hence Exercise 1.43 implies that it is enough to prove the statement of the theorem for the modules $\mathbf{V}^{(n)}$, $n \in \mathbb{N}$.

Assume that $n \in \mathbb{N}$ and the module $\mathbf{V}^{(n)}$ is given by (1.9). Let $(\cdot,\cdot)$ be the symmetric bilinear form on $\mathbf{V}^{(n)}$ which is given by the matrix

$$\begin{pmatrix} 0\,0\,0 \dots 0\,0\,1 \\ 0\,0\,0 \dots 0\,1\,0 \\ 0\,0\,0 \dots 1\,0\,0 \\ \vdots\;\vdots\;\vdots \;\dots\; \vdots\;\vdots\;\vdots \\ 0\,0\,1 \dots 0\,0\,0 \\ 0\,1\,0 \dots 0\,0\,0 \\ 1\,0\,0 \dots 0\,0\,0 \end{pmatrix}$$

in the basis $\{v_0, v_1, \dots, v_{n-1}\}$. From (1.9) it follows by a direct calculation that in this basis the linear operators E, F and H satisfy (1.25). This proves that $\mathbf{V}^{(n)}$ is a $\diamond$-module and completes the proof. $\square$

Note that the proof of Theorem 1.44 can be seen as a kind of justification of the basis $\{v_0, v_1, \dots, v_{n-1}\}$ of the module $\mathbf{V}^{(n)}$.

After the above results, given some $\mathfrak{g}$-module V it is natural to ask how many different forms $(\cdot,\cdot)$ on V the module V is unitarizable for (or a $\diamond$-module). The answer turns out to be easy for simple finite-dimensional modules.

Proposition 1.45. *Let* V *be a simple finite-dimensional* $\mathfrak{g}$*-module. Then the Hermitian inner product with respect to which the module* V *is unitarizable is unique up to a positive real scalar.*

Proof. Let $V \cong \mathbf{V}^{(n)}$, $n \in \mathbf{N}$, and $(\cdot,\cdot)$ be an Hermitian inner product on V with respect to which V is unitarizable. Consider the basis $\{u_0, \dots, u_{n-1}\}$ from the proof of Theorem 1.40. The vectors in this basis are eigenvectors of the self-adjoint linear operator H corresponding to pairwise different eigenvalues. Hence $(u_i, u_j) = 0$ for all $i \neq j$. Let $(u_0, u_0) = c$. Then c is a positive real number. Let us prove that $(u_i, u_i) = c$ for all

$i \in \{0, \dots, n-1\}$ by induction on i. The basis of the induction is trivial. For all $i \in \{1, \dots, n-1\}$ we have

$$\begin{aligned} (u_i, u_i) &\overset{(1.24)}{=} \left(\frac{1}{\sqrt{a_i}} F(u_{i-1}), \frac{1}{\sqrt{a_i}} F(u_{i-1}) \right) \\ \text{(by (1.23))} &= \frac{1}{a_i}(u_{i-1}, E(F(u_{i-1}))) \\ \text{(by (1.24))} &= \frac{1}{a_i}(u_{i-1}, a_i u_{i-1}) \\ &= (u_{i-1}, u_{i-1}) \\ \text{(by induction)} &= c. \end{aligned}$$

The claim follows. □

Proposition 1.46. *Let V be a simple finite-dimensional $\mathfrak{g}$-module. Then the non-degenerate symmetric bilinear form with respect to which the module V is a $\diamond$-module is unique up to a non-zero complex scalar.*

Proof. Let $V \cong \mathbf{V}^{(n)}$, $n \in \mathbf{N}$, and $(\cdot, \cdot)$ be a non-degenerate symmetric bilinear form on V with respect to which V is a $\diamond$-module. Consider the basis $\{v_0, v_1, \dots, v_{n-1}\}$ from (1.9). For $i, j \in \{0, \dots, n-1\}$ by (1.25) we have $(H(v_i), v_j) = -(v_i, H(v_j))$. As all elements of our basis are eigenvectors to H with pairwise different eigenvalues, it follows that $(v_i, v_j) \neq 0$ implies that the eigenvalues λ_i and λ_j of v_i and v_j, respectively, satisfy $\lambda_i = -\lambda_j$. Hence $(v_i, v_j) \neq 0$ implies $i = n-1-j$. Let $(v_0, v_{n-1}) = c$. As $(\cdot, \cdot)$ is non-degenerate, we have $c \neq 0$. Let us show by induction on i that $(v_i, v_{n-1-i}) = c$ for all $i \in \{0, 1, \dots, n-1\}$. For all such $i > 0$ we have

$$\begin{aligned} (v_i, v_{n-1-i}) &\overset{(1.9)}{=} \left(F(v_{i-1}), \frac{1}{a_{n-i}} E(v_{n-i}) \right) \\ \text{(by (1.25))} &= \frac{1}{a_{n-i}}(v_{i-1}, F(E(v_{n-i}))) \\ \text{(by (1.9))} &= \frac{1}{a_{n-i}}(v_{i-1}, a_{n-i} v_{n-i}) \\ &= (v_{i-1}, v_{n-i}) \\ \text{(by induction)} &= c. \end{aligned}$$

The claim follows. □

For a direct sum of simple modules, the description of bilinear forms analogous to Propositions 1.45 and 1.46 will be more complicated. In particular, as an obvious observation one could point out that it is possible to independently rescale the restrictions of the bilinear form to pairwise orthogonal direct summands.

1.6 Bilinear forms on tensor products

Let V and W be two vector spaces and $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ be bilinear forms on V and W respectively. Then the assignment

$$(v \otimes w, v' \otimes w') = (v, v')_1 \cdot (w, w')_2 \tag{1.26}$$

extends to a bilinear form on the tensor product $V \otimes W$.

Exercise 1.47. Check that the form $(\cdot,\cdot)$ is symmetric provided that both $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ are symmetric; that the form $(\cdot,\cdot)$ is non-degenerate provided that both $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ are non-degenerate; and that the form $(\cdot,\cdot)$ is Hermitian provided that both $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ are Hermitian.

Proposition 1.48. *Assume that V and W are unitarizable modules (resp. $\diamond$-modules) for the forms $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ respectively. Then $V \otimes W$ is unitarizable (resp. a $\diamond$-module) for $(\cdot,\cdot)$.*

Proof. We prove the statement for unitarizable modules. For $\diamond$-modules the proof is similar. Due to Exercise 1.47 it is sufficient to check (1.23) for $X \in \{E, F, H\}$. For $v, v' \in V$ and $w, w' \in W$ we have

$$\begin{array}{rcl}
(X(v \otimes w), v' \otimes w') & \overset{(1.17)}{=} & (X(v) \otimes w + v \otimes X(w), v' \otimes w') \\
\text{(by linearity)} & = & (X(v) \otimes w, v' \otimes w') + (v \otimes X(w), v' \otimes w') \\
\text{(by (1.26))} & = & (X(v), v')_1 \cdot (w, w')_2 + (v, v')_1 (X(w), w')_2 \\
\text{(by (1.23))} & = & (v, X^\star(v'))_1 \cdot (w, w')_2 + (v, v')_1 (w, X^\star(w'))_2 \\
\text{(by (1.26))} & = & (v \otimes w, X^\star(v') \otimes w') + (v \otimes w, v' \otimes X^\star(w')) \\
\text{(by linearity)} & = & (v \otimes w, X^\star(v') \otimes w' + v' \otimes X^\star(w')) \\
\text{(by (1.17))} & = & (v \otimes w, X^\star(v' \otimes w')).
\end{array}$$

The claim follows. □

We know that the tensor product of two simple finite-dimensional $\mathfrak{g}$-modules is not simple in general (see Theorem 1.39). Hence the bilinear form making this tensor product module unitarizable or a $\diamond$-module is usually not unique (even up to some scalar). However, we would like to finish this section with a description of one invariant, which turns out in the real case.

Exercise 1.49. Consider the real Lie algebra $\mathfrak{sl}_2(\mathbb{R})$. Show that (1.9) still defines on the real span $\mathbf{V}_{\mathbb{R}}^{(n)}$ of $\{v_0, \dots, v_{n-1}\}$ the structure of a simple $\mathfrak{sl}_2(\mathbb{R})$-module. Check that the analogues of Theorem 1.39 and all the above results from Sections 1.5 and 1.6 are true for $\mathfrak{sl}_2(\mathbb{R})$ with the same proofs.

After Exercise 1.49 one could point out one striking difference between the real versions of Proposition 1.45 and Proposition 1.46. It is the possibility of the sign change in the assertion of Proposition 1.46 (note that two forms which differ by a sign change cannot be obtained from each other by a base change in the original module). Let us call the form on $\mathbf{V}^{(n)}$, described in the proof of Proposition 1.46, *standard*, and the form, obtained from the standard form by multiplying with -1, *non-standard*. Our main result in this section is the following:

Theorem 1.50. *Let $m, n \in \mathbb{N}$, $m \leq n$; $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ be standard forms on $\mathbf{V}_{\mathbb{R}}^{(m)}$ and $\mathbf{V}_{\mathbb{R}}^{(n)}$ respectively; and $(\cdot,\cdot)$ be the form on $\mathbf{V}_{\mathbb{R}}^{(n)} \otimes \mathbf{V}_{\mathbb{R}}^{(m)}$ given by (1.26). Then, up to multiplication with a positive real number, for $i = 0, 1, \dots, m$ the restriction of $(\cdot,\cdot)$ to the direct summand $\mathbf{V}_{\mathbb{R}}^{(n+m-1-2i)}$ of $\mathbf{V}_{\mathbb{R}}^{(n)} \otimes \mathbf{V}_{\mathbb{R}}^{(m)}$ is standard for all even i and non-standard for all odd i.*

Proof. As in the proof of Theorem 1.39 we use induction on m. For $m = 1$ the statement is obvious. To proceed we will need the following lemma:

Lemma 1.51. *Assume that the form $(\cdot,\cdot)'$ on $\mathbf{V}_{\mathbb{R}}^{(n)}$ makes $\mathbf{V}_{\mathbb{R}}^{(n)}$ into a $\diamond$-module. Then $(\cdot,\cdot)'$ is standard if, and only if, $(v_0, F^{n-1}(v_0))' > 0$ and is non-standard if, and only if, $(v_0, F^{n-1}(v_0))' < 0$.*

Proof. From the definition we have that the form $(\cdot,\cdot)'$ is standard if, and only if, $(v_0, v_{n-1})' > 0$. From (1.9) we have $F^{n-1}(v_0) = v_{n-1}$. The claim follows. $\square$

Let $m = 2$, $n \geq 2$ and assume that $\mathbf{V}_{\mathbb{R}}^{(n)}$ is given by (1.9) and $\mathbf{V}_{\mathbb{R}}^{(2)}$ is given by (1.19). As all coefficients in (1.17) are positive, we get that $F^n(v_0 \otimes e_1) = c\, v_{n-1} \otimes e_2$, where $c > 0$. As

$$(v_0 \otimes e_1, v_{n-1} \otimes e_2) = (v_0, v_{n-1})_1 (e_1, e_2)_2 = 1 > 0$$

(here we used that both $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ are standard), from Lemma 1.51 we obtain that the restriction of $(\cdot,\cdot)$ to the direct summand $\mathbf{V}_{\mathbb{R}}^{(n+1)}$ of $\mathbf{V}_{\mathbb{R}}^{(n)} \otimes \mathbf{V}_{\mathbb{R}}^{(2)}$ is standard.

For the element $w = v_1 \otimes e_1 - (n-1)v_0 \otimes e_2 \neq 0$ we have $E(w) = 0$, so w generates the direct summand $\mathbf{V}_{\mathbb{R}}^{(n-1)}$ of $\mathbf{V}_{\mathbb{R}}^{(n)} \otimes \mathbf{V}_{\mathbb{R}}^{(2)}$. A direct computation shows that

$$F^{n-2}(w) = v_{n-1} \otimes e_1 - v_{n-2} \otimes e_2.$$

Another direct computation then shows that

$$(v_1 \otimes e_1 - (n-1)v_0 \otimes e_2, v_{n-1} \otimes e_1 - v_{n-2} \otimes e_2) = -n < 0.$$

Hence from Lemma 1.51 we obtain that the restriction of $(\cdot,\cdot)$ to the direct summand $\mathbf{V}_{\mathbb{R}}^{(n+1)}$ of $\mathbf{V}_{\mathbb{R}}^{(n)} \otimes \mathbf{V}_{\mathbb{R}}^{(2)}$ is non-standard. This completes the proof of the theorem in the case $m = 2$.

For $k \in \mathbb{N}$ let us denote $\mathbf{V}_{\mathbb{R}}^{(k,+)}$ and $\mathbf{V}_{\mathbb{R}}^{(k,-)}$ the module $\mathbf{V}_{\mathbb{R}}^{(k)}$ endowed with a standard and non-standard (up to a positive real scalar) form, respectively. Then we have just proved that

$$\mathbf{V}_{\mathbb{R}}^{(n,+)} \otimes \mathbf{V}_{\mathbb{R}}^{(2,+)} \cong \mathbf{V}_{\mathbb{R}}^{(n+1,+)} \oplus \mathbf{V}_{\mathbb{R}}^{(n-1,-)}. \tag{1.27}$$

Note that we obviously have

$$\mathbf{V}_{\mathbb{R}}^{(n,+)} \otimes \mathbf{V}_{\mathbb{R}}^{(1,+)} \cong \mathbf{V}_{\mathbb{R}}^{(n,+)}, \quad \mathbf{V}_{\mathbb{R}}^{(n,+)} \otimes \mathbf{V}_{\mathbb{R}}^{(1,-)} \cong \mathbf{V}_{\mathbb{R}}^{(n,-)}. \tag{1.28}$$

In this notation the statement of our theorem can be written as follows:

$$\mathbf{V}_{\mathbb{R}}^{(n,+)} \otimes \mathbf{V}_{\mathbb{R}}^{(m,+)} \cong \mathbf{V}_{\mathbb{R}}^{(n+m-1,+)} \oplus \mathbf{V}_{\mathbb{R}}^{(n+m-3,-)} \oplus \mathbf{V}_{\mathbb{R}}^{(n+m-5,+)} \oplus \dots.$$

The induction step now follows using (1.27) and (1.28) and, rewriting in this new notation, the calculations in (1.21) and (1.22). We leave the details to the reader. □

1.7 Addenda and comments

1.7.1

Alternative expositions for the material presented in Sections 1.1–1.4 can be found in a large number of books and articles, see for example [37, 46, 49, 57, 106]. Many of the results are true or have analogs in much more general contexts (which also can be found in the books listed above). In particular, simple finite-dimensional modules are classified (see Theorem 1.22) and Weyl's Theorem (Theorem 1.29) is true for all simple finite-dimensional complex Lie algebras. For all such algebras there is also an analog of Theorem 1.39, however its formulation is more complicated, as higher multiplicities appear on the right hand side.

1.7.2

If A is an associative algebra with associative multiplication $\cdot$, then one can define on A the structure of a Lie algebra using the operation of taking the commutator with respect to $\cdot$: $[a,b] = a \cdot b - b \cdot a$. The Lie algebra $(A, [\cdot,\cdot])$ is called the Lie algebra *underlying* the associative algebra $(A,\cdot)$ and is often denoted by $A^{(-)}$. In particular, if V is a vector space, one can consider the associative algebra $\mathcal{L}(V)$ of all linear operators on V and the underlying Lie algebra $\mathcal{L}(V)^{(-)}$.

An $\mathfrak{sl}_2$-module is then given by a Lie algebra *homomorphism* from $\mathfrak{sl}_2$ to $\mathcal{L}(V)^{(-)}$, that is a linear map $\varphi : \mathfrak{sl}_2 \to \mathcal{L}(V)$, which satisfies

$$\varphi([x,y]) = [\varphi(x), \varphi(y)] \tag{1.29}$$

for all $x, y \in \mathfrak{sl}_2$. For such φ in the notation of Section 1.1 we simply have $H = \varphi(\mathbf{h})$, $F = \varphi(\mathbf{f})$ and $E = \varphi(\mathbf{e})$.

Substituting $\mathfrak{sl}_2$ with an arbitrary Lie algebra, one obtains the notion of a *module* over any Lie algebra. The homomorphism φ is usually called a *representation* of the Lie algebra. Hence the notions of module and representation are equivalent, differing only in their emphasis on the underlying vector space V (for modules) or the homomorphism φ (for representations). Sometimes one can also say that a representation defines an *action* of the Lie algebra on the underlying vector space V.

1.7.3

The Lie algebra $\mathfrak{sl}_2$ is a subalgebra of the Lie algebra $\mathfrak{gl}_2$, the latter being the underlying Lie algebra of the associative algebra of all complex 2×2 matrices. Moreover, the algebra $\mathfrak{gl}_2$ is a direct sum of $\mathfrak{sl}_2$ and the commutative Lie subalgebra of all scalar matrices.

1.7.4

Weyl's Theorem can be proved using the notion of unitarizability of finite-dimensional modules. Let V be an arbitrary finite-dimensional $\mathfrak{g}$-module. Using the exponential map one first could lift the $\mathfrak{g}$-action on V to the action of the group SL(2) and further SU(2). In fact one can show that there is a natural bijection between finite-dimensional $\mathfrak{g}$-modules, finite-dimensional SL(2)-modules and finite-dimensional SU(2)-modules. As SU(2) is compact, all finite-dimensional SU(2)-modules and hence all finite-dimensional

$\mathfrak{g}$-module are completely reducible. We refer the reader to Chapter III, §6 of [106] for details.

1.7.5

So far, all modules which we considered were *left* modules. There is also the natural notion of a *right* module. The triple E', F' and H' of linear operators on a vector space V defines on V the structure of a *right* $\mathfrak{g}$-module provided that the operators E', F' and H' satisfy

$$F'E' - E'F' = H', \quad E'H' - H'E' = 2E', \quad F'H' - H'F' = -2F'.$$

This corresponds to an *antihomomorphism* from $\mathfrak{g}$ to $\mathcal{L}(V)^{(-)}$, that is to a linear map $\varphi : \mathfrak{g} \to \mathcal{L}(V)$, which satisfies

$$\varphi([x, y]) = [\varphi(y), \varphi(x)] \tag{1.30}$$

for all $x, y \in \mathfrak{sl}_2$. In what follows, by "module" we will always mean a left module.

1.7.6

If A is an algebra (associative or Lie), then for any left A-module V the dual space $V^* = \mathrm{Hom}(V, \mathbb{C})$ carries the natural structure of a right A-module given by $(a(f))(v) = f(a(v))$ for $a \in A$, $v \in V$ and $f \in V^*$.

However, if A is an algebra (associative or Lie) with a fixed anti-involution $\natural$, then for any left A-module V the space V^* carries the natural structure of a left A-module given by $(a(f))(v) = f(a^\natural(v))$ for $a \in A$, $v \in V$ and $f \in V^*$.

Any element $\varphi \in \mathrm{Hom}_A(V, V^*)$ defines a bilinear form on V as follows: $(v, w)_\varphi = \varphi(v)(w)$. This form obviously satisfies

$$(a(v), w) = (v, a^\natural(w)). \tag{1.31}$$

If V is finite-dimensional then every bilinear form on V satisfying (1.31) has the form $(v, w)_\varphi$ for some $\varphi \in \mathrm{Hom}_A(V, V^*)$. The form $(v, w)_\varphi$ is non-degenerate if, and only if, φ is an isomorphism. These general arguments give an alternative proof of Propositions 1.45 and 1.46. More details and some further related results can be found in [93].

1.7.7

Theorem 1.50 appears in [6] (in the form presented later on in Exercise 1.70) in connection to the study of Hodge–Riemann relations for polytopes. Our proof follows the general idea of [6]. There exists an alternative "brute force" argument for Theorem 1.50 worked out in [71]. Here is its outline:

Let $m, n \in \mathbb{N}$ and $n \geq m$. Assume that $\mathbf{V}^{(n)}$ is given by (1.9) and $\mathbf{V}^{(m)}$ is similarly given by (1.9) in the basis $\{w_0, w_1, \dots, w_{m-1}\}$. For $k = 0, 1, \dots, m-1$ set $l_k = m + n - 2 - 2k$. A direct calculation shows that the element

$$u_k = \sum_{i=0}^{k} (-1)^i v_{n-1+i} \otimes w_{m-1+k-i}$$

satisfies

$$H(u_k) = -l_k u_k, \quad F(u_k) = 0.$$

In the same way as in Lemma 1.51, one shows that the form $(\cdot,\cdot)'$ on $\mathbf{V}^{(n)}_{\mathbb{R}}$, making $\mathbf{V}^{(n)}_{\mathbb{R}}$ into a $\diamond$-module, is standard if, and only if, $(v_{n-1}, E^{n-1}(v_{n-1}))' > 0$ and non-standard if, and only if, $(v_{n-1}, E^{n-1}(v_{n-1}))' < 0$. Hence to prove Theorem 1.50 one simply has to show that for any $k = 0, 1, \dots, m-1$ the sign of the number $(u_k, E^{l_k}(u_k))$ alternates with k. This reduces to the computation of $E^{l_k}(v_k)$, which is not entirely straightforward. However, using the following *Karlsson–Minton identity* for the hypergeometric series ${}_3F_2$: for all integers a, b, c, d such that $0 \leq b \leq a \leq \min(c, d)$ we have

$$\sum_{i=\max(0,2a-b-d)}^{\min(a,c-b)} (-1)^i \frac{(c-i)!(d-a+i)!}{i!(a-i)!(c-b-i)!(d+b-2a+i)!} = (-1)^{a+b},$$

one proves by a direct calculation that

$$E^{l_k}(u_k) = l_k! \sum_{i=0}^{k} (-1)^{k+i} \frac{(m-1-i)!\,(n-1-k+i)!}{i!\,(k-i)!} v_{k-i} \otimes w_i.$$

Using the last formula, the computation of $(u_k, E^{n+m-2-2k}(u_k))$ is fairly straightforward and yields the necessary result. We refer the reader to [71] for details.

1.8 Additional exercises

Exercise 1.52. Let $\mathfrak{a}$ denote the vector space with the basis $\{e_{-1}, e_0, e_1\}$. Define the bilinear operation $[\cdot,\cdot]$ on $\mathfrak{a}$ via

$$[e_i, e_j] = \begin{cases} (j-i)e_{i+j}, & i + j \in \{-1, 0, 1\}; \\ 0, & \text{otherwise}. \end{cases}$$

Show that this makes $\mathfrak{a}$ into a Lie algebra. Show further that $\mathfrak{a}$ is isomorphic to $\mathfrak{sl}_2$.

Exercise 1.53. Let $\mathfrak{b}$ denote the vector space with the basis $\{a, b, c\}$. Define the antisymmetric bilinear operation $[\cdot, \cdot]$ on $\mathfrak{b}$ via

$$[a, b] = c, \quad [b, c] = a, \quad [c, a] = b.$$

Show that this makes $\mathfrak{b}$ into a Lie algebra. Show further that $\mathfrak{b}$ is isomorphic to $\mathfrak{sl}_2$.

Exercise 1.54. Consider the vector space $V = \mathbb{C}[x, y]$ and the linear operators

$$E = x \cdot \frac{\partial}{\partial y}, \quad F = y \cdot \frac{\partial}{\partial x}, \quad H = x \cdot \frac{\partial}{\partial x} - y \cdot \frac{\partial}{\partial y}$$

on V.

(a) Show that the operators E, F and H make V into a $\mathfrak{g}$-module.
(b) Show that for every $n \in \mathbb{N}_0$ the linear span of all homogeneous polynomials of degree n is a submodule of V, isomorphic to $\mathbf{V}^{(n+1)}$.

Exercise 1.55. Consider the vector space $V = \mathrm{Mat}_{3\times 3}(\mathbb{C})$ of all complex 3×3 matrices and the matrices

$$X = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Show that the linear operators

$$E = [X, \_], \quad F = [Y, \_], \quad H = [Z, \_]$$

(here $E(A) = [X, A] = XA - AX$ for all $A \in \mathrm{Mat}_{3\times 3}(\mathbb{C})$ and similarly for F and H) define on V the structure of a $\mathfrak{g}$-module and determine its decomposition into a direct sum of $\mathbf{V}^{(n)}$'s.

Exercise 1.56. Let $n \in \mathbb{N}$. For every $\lambda \in \{-n+1, -n+3, \dots, n-3, n-1\}$ fix some non-zero element $x_\lambda \in \mathbf{V}^{(n)}_\lambda$ (see (1.8)). Show that the set

$$\mathbf{x} = \{x_\lambda \, : \, \lambda \in \{-n+1, -n+3, \dots, n-3, n-1\}\}$$

is a basis of $\mathbf{V}^{(n)}$.

Exercise 1.57 (Gelfand–Zetlin model, [52]). Let $c \in \mathbb{C}$ be a fixed complex number. For $n \in \mathbb{N}$ set $c' = c - n$ and consider the set $\mathbf{T}_{c,n}$ consisting of all tableaux

$$\mathfrak{t}_c(a) = \begin{array}{l} \boxed{c}\boxed{c'} \\ \;\;\boxed{a} \end{array}$$

where $a \in \{c - i : i = 0, 1, \ldots, n-1\}$. Let $V = V_{c,n}$ denote the linear span of all elements from $\mathbf{T}_{c,n}$. Define the linear operators E, F and H on V as follows:

$$F(\mathfrak{t}_c(a)) = \begin{cases} \mathfrak{t}_c(a-1), & \mathfrak{t}_c(a-1) \in \mathbf{T}_{c,n}; \\ 0, & \text{otherwise.} \end{cases}$$

$$E(\mathfrak{t}_c(a)) = \begin{cases} -(c-a)(c'-a)\mathfrak{t}_c(a+1), & \mathfrak{t}_c(a+1) \in \mathbf{T}_{c,n}; \\ 0, & \text{otherwise.} \end{cases}$$

$$H(\mathfrak{t}_c(a)) = (2a - c - c' - 1) \cdot \mathfrak{t}_c(a).$$

Show that this turns V into a $\mathfrak{g}$-module, which is isomorphic to $\mathbf{V}^{(n)}$.

Exercise 1.58. Write a Cayley table of the Lie algebra $\mathfrak{gl}_2$ in the standard basis $\{e_{11}, e_{12}, e_{21}, e_{22}\}$ consisting of matrix units.

Exercise 1.59 (Gelfand–Zetlin model for $\mathfrak{gl}_2$, [52]).
(a) Show that the $\mathfrak{sl}_2$-module structure on the module $V_{c,n}$ from Exercise 1.57 can be extended to a $\mathfrak{gl}_2$-module structure in the following way:

$$\begin{aligned} e_{12}(\mathfrak{t}_c(a)) &= E(\mathfrak{t}_c(a)), \\ e_{21}(\mathfrak{t}_c(a)) &= F(\mathfrak{t}_c(a)), \\ e_{11}(\mathfrak{t}_c(a)) &= a \cdot \mathfrak{t}_c(a), \\ e_{22}(\mathfrak{t}_c(a)) &= (c + c' - a + 1) \cdot \mathfrak{t}_c(a). \end{aligned}$$

(b) Show that the module $V_{c,n}$ is a simple $\mathfrak{gl}_2$-module.
(c) Show that $V_{c,n} \cong V_{d,m}$ if, and only if, $c = d$ and $n = m$.
(d) Show that every simple $\mathfrak{gl}_2$-module is isomorphic to $V_{c,n}$ for some $c \in \mathbb{C}$ and $n \in \mathbb{N}$.

Exercise 1.60. Construct a counterexample which shows that Weyl's theorem fails for finite-dimensional $\mathfrak{gl}_2$-modules.

Exercise 1.61. Let A and B be two linear operators on some finite-dimensional vector space V.

(a) Prove that $[A, B] = \lambda \cdot A$ for some $\lambda \in \mathbb{C}$, $\lambda \neq 0$, implies that the operator A is nilpotent.
(b) Prove that $[A, B] = A^2$ implies that the operator A is nilpotent.
(c) Prove that $[A, [A, B]] = 0$ implies that the operator $[A, B]$ is nilpotent.

Exercise 1.62. Let V be a $\mathfrak{g}$-module and $n \in \mathbb{N}$. Consider the n-th tensor power $V^{\otimes n}$ of V. Let $\mathbf{S}_n$ denote the symmetric group on $\{1, 2, \dots, n\}$. Show that the linear span of all vectors of the form

$$\sum_{\sigma \in S_n} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n-1)} \otimes v_{\sigma(n)},$$

where $v_1, v_2, \dots, v_n \in V$ forms a $\mathfrak{g}$-submodule of $V^{\otimes n}$. This submodule is called the n-th *symmetric power* of V and is denoted by $\mathrm{Sym}^n(V)$.

Exercise 1.63. Prove that $\mathrm{Sym}^n(\mathbf{V}^{(2)}) \cong \mathbf{V}^{(n+1)}$.

Exercise 1.64. Let V and $n \in \mathbb{N}$ be as in Exercise 1.62. Show that the linear span of all vectors of the form

$$\sum_{\sigma \in S_n} \mathrm{sign}(\sigma) \cdot v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n-1)} \otimes v_{\sigma(n)},$$

where $v_1, v_2, \dots, v_n \in V$ forms a $\mathfrak{g}$-submodule of V. This submodule is called the n-th *exterior power* of V and is denoted by $\bigwedge^n(V)$.

Exercise 1.65. Prove that $\bigwedge^n(\mathbf{V}^{(n)}) \cong \mathbf{V}^{(1)}$.

Exercise 1.66. Prove that $\bigwedge^n(V) \cong 0$ provided that $n > \dim V$.

Exercise 1.67.

(a) Show that the correspondence

$$\mathbf{e}^\natural = -\mathbf{e}, \quad \mathbf{f}^\natural = -\mathbf{f}, \quad \mathbf{h}^\natural = -\mathbf{h}$$

uniquely extends to an anti-involution $\natural$ on $\mathfrak{g}$.
(b) Prove that for every $n \in \mathbb{N}$ there exists a unique (up to a non-zero scalar) non-degenerate bilinear form $(\cdot, \cdot)_n$ on $\mathbf{V}^{(n)}$ such that

$$(X(v), w)_n = (v, X^\natural(w))_n$$

for all $X \in \{\pm E, \pm F, \pm H\}$ and all $v, w \in \mathbf{V}^{(n)}$.
(c) Prove that the form $(\cdot, \cdot)_n$ is symmetric for odd n and antisymmetric for even n.

Exercise 1.68. Formulate and prove an analogue of Proposition 1.48 for the anti-involution $\natural$ from Exercise 1.67.

Exercise 1.69. Formulate and prove an analog of Theorem 1.50 for the form $(\cdot,\cdot)_n$ from Exercise 1.67.

Exercise 1.70 ([6]). For $n \in \mathbb{N}$ set

$$\tilde{\mathbf{V}}^{(n)} = \begin{cases} \mathbf{V}^{(n,+)}, & n \text{ is odd;} \\ \mathbf{V}^{(n,-)}, & n \text{ is even} \end{cases}$$

(see notation of Theorem 1.50). Show that for any $k, m \in \mathbb{N}$ the tensor product $\tilde{\mathbf{V}}^{(k)} \otimes \tilde{\mathbf{V}}^{(m)}$ decomposes into a direct sum of modules of the form $\tilde{\mathbf{V}}^{(n)}$, $n \in \mathbb{N}$.

Exercise 1.71. Fix $n \in \mathbb{N}$. For $i = 1, \dots, n-1$ let $\mathtt{s}_i$ denote the involution $(i, i+1)$ of the symmetric group $\mathbf{S}_n$. The group $\mathbf{S}_n$ is a Coxeter group in the natural way. Denote by R the set of all reflections in $\mathbf{S}_n$. In particular, the $\mathtt{s}_i$'s are simple reflections. Denote by $\mathtt{e}$ the identity element of $\mathbf{S}_n$. For $i = 0, \dots, n-1$ and $j = 0, \dots, i$ set

$$\mathtt{x}_{i,j} = \begin{cases} \mathtt{e}, & j = 0; \\ \mathtt{s}_i \mathtt{s}_{i-1} \dots \mathtt{s}_{i-j+1}, & j > 0; \end{cases}$$

and define $X_i = \{\mathtt{x}_{i,0}, \mathtt{x}_{i,1}, \dots, \mathtt{x}_{i,i-1}\}$. Let $\mathtt{X}_i$ denote the formal linear span of the elements from X_i. Define on $\mathtt{X}_i$ the structure of a $\mathfrak{g}$-module via (1.9) using the convention $v_j = \mathtt{x}_{i,i-1-j}$.

(a) Show that every $\alpha \in \mathbf{S}_n$ admits a unique decomposition of the form $\alpha = \alpha_0 \alpha_1 \dots \alpha_{n-1}$, where $\alpha_i \in X_i$ for all i.

(b) Show that the underlying space of the tensor product $\mathtt{X}_0 \otimes \mathtt{X}_1 \otimes \dots \otimes \mathtt{X}_{n-1}$ can be canonically identified with $\mathbb{C}[\mathbf{S}_n]$ via the map

$$\mathtt{x}_{0,j_0} \otimes \mathtt{x}_{1,j_1} \otimes \dots \otimes \mathtt{x}_{n-1,j_{n-1}} \mapsto \mathtt{x}_{0,j_0} \mathtt{x}_{1,j_1} \cdots \mathtt{x}_{n-1,j_{n-1}}.$$

This equips $\mathbb{C}[\mathbf{S}_n]$ with the structure of a $\mathfrak{g}$-module. (This structure comes from the Hard Lefschetz theorem (see [53]) applied to the cohomology algebra of the flag variety, which can be naturally identified with $\mathbb{C}[\mathbf{S}_n]$ as a vector space, see [56].)

(c) Show that every $\alpha \in \mathbf{S}_n$ is a eigenvector for $\mathbf{h}$ (with respect to the $\mathfrak{g}$-module structure described in (b)) with the eigenvalue

$$|\{r \in R : r\alpha < \alpha\}| - |\{r \in R : r\alpha > \alpha\}|,$$

where $<$ denotes the Bruhat order on $\mathbf{S}_n$. Show that the same number equals $-\frac{n(n-1)}{2} + 2l(\alpha)$, where $l(\alpha)$ is the length of α with respect to the set $\{\mathtt{s}_1, \mathtt{s}_2, \dots \mathtt{s}_{n-1}\}$ of simple reflections.

Exercise 1.72. Generalize Exercise 1.71 to other Coxeter groups.

Exercise 1.73. For every finite-dimensional $\mathfrak{g}$-module V we define the function $\mathrm{ch}_V : \mathbb{Z} \to \mathbb{N}_0$ as follows:

$$\mathrm{ch}_V(\lambda) = \dim V_\lambda, \quad \lambda \in \mathbb{Z}.$$

(a) Show that $\mathrm{ch}_V(\lambda) = 0$ for all $\lambda \in \mathbb{Z}$ such that $|\lambda|$ is big enough.
(b) Show that $\mathrm{ch}_V(\lambda) = \mathrm{ch}_V(-\lambda)$ for all $\lambda \in \mathbb{Z}$.
(c) Show that $\mathrm{ch}_V(\lambda) \geq \mathrm{ch}_V(\mu)$ for all elements $\lambda, \mu \in \mathbb{Z}$ of the same parity such that $0 \leq |\lambda| \leq |\mu|$.
(d) Show that $\mathrm{ch}_V = \mathrm{ch}_W$ if and only if $V \cong W$.
(e) Show that for any function $\mathrm{ch} : \mathbb{Z} \to \mathbb{N}_0$, which has the properties, described in (a)–(c) above, there exists a unique (up to isomorphism) $\mathfrak{g}$-module V such that $\mathrm{ch} = \mathrm{ch}_V$.

Exercise 1.74 ([60]). Show that the elements $\mathbf{x} = \mathbf{h}$, $\mathbf{y} = 2\mathbf{e} - \mathbf{h}$ and $\mathbf{z} = -2\mathbf{f} - \mathbf{h}$ form a basis of $\mathfrak{g}$ and that we have

$$[\mathbf{x}, \mathbf{y}] = 2\mathbf{x} + 2\mathbf{y}, \quad [\mathbf{y}, \mathbf{z}] = 2\mathbf{y} + 2\mathbf{z}, \quad [\mathbf{z}, \mathbf{x}] = 2\mathbf{z} + 2\mathbf{x}$$

The basis $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is called the *equitable* basis of $\mathfrak{g}$.