Chapter 1

Mathematics Before Leonhard Euler

> "Number rules the Universe."
>
> *The Pythagoreans*

> "Geometry has two great treasures: one is the theorem of Pythagoras; the other, the division of a line in extreme and mean ratio. The first we may name as a measure of gold, the second we may name as a precious jewel."
>
> *Johann Kepler*

> "As long as algebra and geometry proceed along separate paths, their advance was slow and their applications were limited. But when these sciences joined company, they drew from each other fresh vitality and hence forward marched on at a rapid pace towards perfection."
>
> *Joseph Louis Lagrange*

1.1 Introduction

Historically, mathematics originated from the fundamentals of counting in arithmetic. It is considered one of the greatest achievements of the human endeavor. Originally, it was the study of numbers or symbols and their relations. These symbols were created to stand for the *natural numbers* 1, 2, 3, $\cdots$ which form an infinite collection on which the basic arithmetic operations of addition and multiplication could be performed. It was the Ancient Hindus and Greeks who first discovered the natural numbers, but they did not acknowledge negative numbers. The first systematic algebra to use zero, negative numbers, and the decimal system was developed by

Hindu mathematicians in India during the seventh century A.D. They used positive and negative numbers to handle financial transactions involving credit and debit. Subsequently, mathematics has successfully been used to precisely formulate laws of nature.

Mathematics has more than 5000 years of history. By 3000 B.C., the people of Babylonia, China, Egypt and India had developed early and practical number systems. They used the knowledge of number systems in business, industry, government, science and indeed, in everyday life. Between 600 and 300 B.C., the Greeks took the next great step in advancing the knowledge of arithmetic, algebra, geometry and astronomy. They appear to have been the first to develop mathematical theory of arithmetic and geometry. Subsequently, it was realized that all scientific problems depend on mathematics for qualitative and quantitative descriptions and mathematical formulas became very useful for experiments and observations.

1.2 Pythagoras, the Pythagorean School and Euclid

Most of our knowledge of mathematics of the classical age came from the writings of many mathematicians and philosophers including Pythagoras (580-500 B.C.), Euclid (330-275 B.C.), Archimedes (287-212 B.C.) and Apollonius (260-200 B.C.). For the Greeks, mathematics was then largely synonymous with geometry which dealt with the measurement of land. Indeed, geometry was derived from two Greek words meaning *measurement of the earth*. The Ancient Egyptians used geometry to measure the size of their firm lands, and to find boundaries of these firm lands after yearly floods of the Nile River washed away or covered old landmarks. Classically, geometry dealt with the size, shape, area, volume or position of any object. More importantly, geometrical concepts and numerical ideas have been wrapped up together for thousands of years and they cannot be separated at all.

In about 540 B.C., Pythagoras established a school of mathematics and natural philosophy at Crotona in southern Italy. The influence of this great master Pythagoras was simply remarkable as his students and followers were very loyal to him and they formed themselves a society or brotherhood. They were known as the Order of the Pythagoreans. Members of the Pythagorean School were very obedient and loyal to their great master, shared everything in common, held the same religious and philosophical beliefs, made a commitment to the same pursuits and bound themselves to an oath not to reveal their own secrets and teachings of the school.

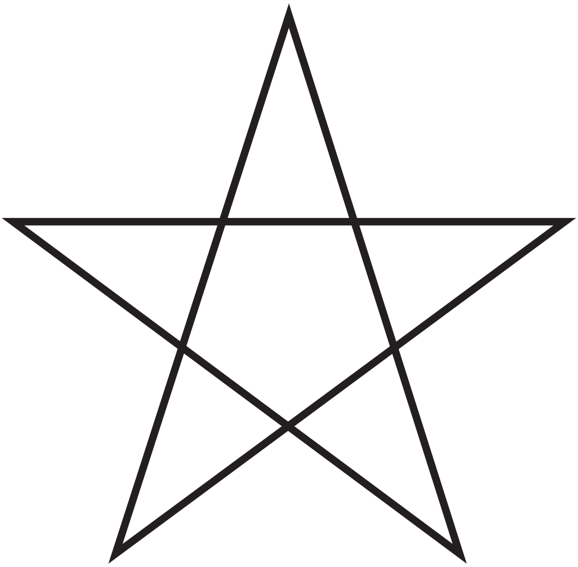

Fig. 1.1 The star pentagon.

It is remarkable that the school discovered a beautiful *star pentagon* or *pentagram* (see Figure 1.1), the most fitting badge of the Pythagorean brotherhood. It was also a fitting symbol of mathematics and the Greek emblem of health. In addition to their unique contributions to mathematics, particularly, to geometry and number theory; the Pythagoreans were specially interested in the study of medicine and music. Figure 1.2 shows an infinite sequence of nested pentagons.

They developed a large body of knowledge in geometry and properties of numbers, and proved a large number of geometrical theorems including one of the most famous theorems in geometry known as the *Pythagorean Theorem*:

$$c^2 = a^2 + b^2 \tag{1.2.1}$$

for any right-angled triangle of sides a and b adjoining the right angle and c is the hypotenuse.

This theorem has probably received more diverse proofs than any other theorem in all of mathematics. In the second edition of his book entitled The Pyathgorean Proposition, E. S. Loomis (1968) has reported about 367 demonstrations (or proofs) of this famous theorem. Making reference to Figure 1.3, a dissection type proof of this famous theorem can be given as

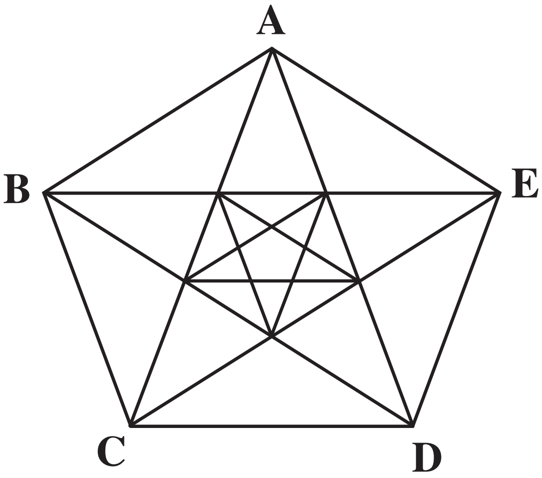

Fig. 1.2 Sequence of nested pentagons.

follows. The first square of side $(a + b)$ is dissected into four equal right angled triangles of sides a and b and a square of side c so that $(a + b)^2 = 4(\frac{1}{2}ab) + c^2 = 2ab + c^2$. The second figure is dissected into two squares and four equal right angled triangles so that $(a + b)^2 = 4(\frac{1}{2}ab) + a^2 + b^2$. Equating two equal expressions readily gives (1.2.1).

One of the Indian mathematicians, Bhaskara gave a second proof of the Pythagorean theorem by drawing the altitude on the hypotenuse of the

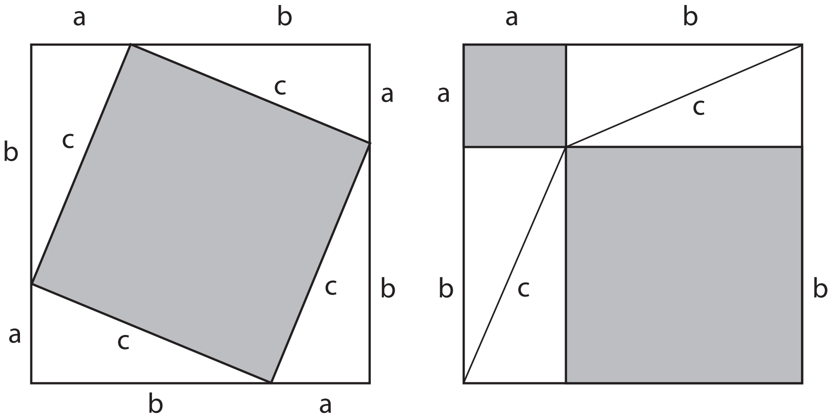

Fig. 1.3 Dissection of two equal squares.

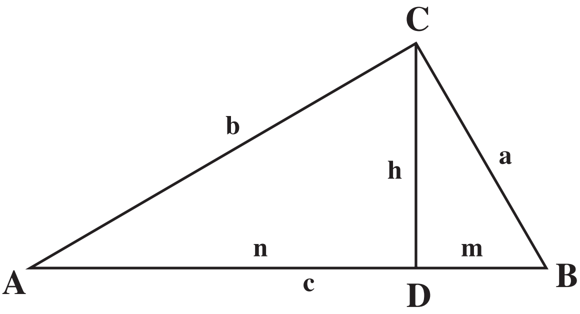

Fig. 1.4 A right angled triangle with $\angle ACB = 90°$.

right angled triangle ABC with $\angle ACB = 90°$. It follows from similar right angled triangles as shown in Figure 1.4 that

$$\frac{a}{c} = \frac{m}{a} \quad \text{and} \quad \frac{b}{c} = \frac{n}{b}.$$

Bhaskara gave another proof using dissection in which the square on the hypotenuse is divided into four equal triangles, (see Figure 1.5) each is congruent to the given right angled triangle of sides a, b, and c and a square with side $b - a$. Clearly, a simple algebra supplies the proof as follows:

$$c^2 = 4\left(\frac{1}{2}ab\right) + (b-a)^2 = a^2 + b^2$$

or

$$a^2 = cm \quad \text{and} \quad b^2 = cn$$

so that

$$a^2 + b^2 = c(m+n) = c^2.$$

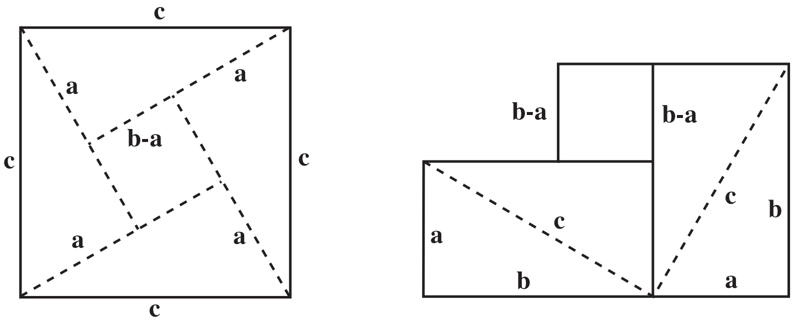

Fig. 1.5 Dissection of a square into four triangles and a square.

A famous British mathematician, John Wallis (1616-1703) rediscovered this ancient proof in the seventeenth century. For several other proofs of the Pythagorean theorem, the reader is referred to Loomis' book (1940) and to a book entitled *Great Moments in Mathematics Before 1650* by Howard Eves (1911-2004) published in 1983.

When $a = b = 1$, $c = \sqrt{2}$, an *irrational number* which cannot be expressed as the ratio of two integers. In other words, the rational numbers are not adequate for measuring the hypothenuse of a right-angled triangle whose base and height are unity. The discoveries of Pythagoras theorem and the irrational numbers were the greatest achievements of the Pythagoreans in the history of mathematics. Indeed, the Pythagorean discovery of an irrational number led to the solution of equations such as

$$x^2 = 2. \tag{1.2.2}$$

Although $x = \sqrt{2}$ is irrational, but it can be expressed in terms of approximate rational numbers 1.4, 1.41, 1.414, $\cdots$ with finite number of decimal places.

The Pythagoreans also proved many geometrical theorems including the equality of the base angles of an isosceles triangle, and the sum of three angles of a triangle is equal to two right angle. They also proved the famous algebraic identities

$$(a \pm b)^2 = a^2 \pm 2ab + b^2 \tag{1.2.3ab}$$

using purely geometrical arguments.

More remarkably, they made three great discoveries: the first, one was the introduction of *proof* in mathematics, that mathematical proof must proceed from given assumptions, the second one was that the natural numbers were insufficient for the construction of mathematics, and the third one was that the set of natural, rational and irrational numbers form the complete set of real numbers with the geometrical interpretation. Geometrically, to each real number corresponds to one and only one point on the real line. In addition, there were three famous unsolved problems that exerted so great influence on the development of Greek mathematics. The original idea was to solve them by *ruler and compasses* constructions. However, the impossibility of solutions by a ruler and a compass kept these problems at the center of the mathematical stage for many centuries.

The first problem was known as the *Delian problem* which dealt with the doubling of a cube, that is, to construct a cube whose volume is twice that of a given cube. Mathematically, the problem is to find a solution of

$$x^3 = 2. \tag{1.2.4}$$

In about 400 B.C., Archytas brilliantly solved the problem by finding the point of intersection of three surfaces in three-dimensional space: a cylinder, a cone, and a torus generated by rotating a circle about one of its tangents. This was indeed a most remarkable achievement of Archytas as there was little known then about three-dimensional (or solid) geometry.

The second problem was the trisection of a given angle θ by a ruler and a compass. Mathematically, it reduces to a solution for θ which satisfies the equation

$$4x^3 - 3x - \cos\theta = 0. \tag{1.2.5}$$

For $\theta = 60^\circ$ so that $\cos\theta = \frac{1}{2}$. The polynomial on the left of (1.2.5) is irreducible over the field Q of rational numbers. It can be shown that θ cannot belong to a field extension E of Q of degree 2^m. Consequently, the trisection of an angle $\theta = 60^\circ$ is *not* possible with a ruler and a compass. For the construction of regular polygons with a ruler and a compass, the set of complex solution of the well-known *cyclotomic equation*

$$x^n - 1 = 0 \tag{1.2.6}$$

contains the number one and divides the unit circle into n equal parts. The solution is possible with the aid of the following theorem due to Gauss:

Theorem of Gauss: A regular n-gon can be constructed with ruler and compass if and only if

$$n = 2^m p_1 p_2 \cdots p_r, \tag{1.2.7}$$

where m is a natural number and $p_r's$ are pair distinct Fermat's primes of the form

$$F_k = 2^{2^k} + 1, \qquad k = 0, 1, 2, \cdots. \tag{1.2.8}$$

It is probably known that for $k = 0$, 1, 2, 3, 4, the above number is prime. Consequently, a regular n-gon can be constructed for n in the list of primes 2, 3, 5, 17, 257, $65,537$. For $n \leq 20$, the construction of all regular n-gons with $n = 3$, 4, 5, 6, 8, 10, 12, 15, 16, 17,20 is possible using only a ruler and a compass.

Finally, the third problem was to square the circle, that is, to construct a square with ruler and compass whose area is equal to that of a given unit circle. The length x for the sides of a square is a solution of the equation

$$x^2 = \pi. \tag{1.2.9}$$

In 1882, Ferdinand Lindemann (1852-1939), David Hilbert's (1862-1943) teacher, proved the transcendence of the number π over the field Q of rational numbers. Consequently, number x or π cannot be an element of any algebraic field extension of Q. So the problem has no solution. Clearly, the first two problems are algebraic so that they require the solution of a cubic equation. The third problem is totally different as it involves the transcendental number π.

Indeed, in mathematics, the Pythagoreans made great progress, particularly in the theory of numbers, and in geometry of line, plane and solid figures, and also, lengths, areas, and volumes associated with them. It is the most appropriate to recall the delightful quotation of Johann Kepler (1571-1630): "Geometry has two great treasures: one is the theorem of Pythagoras; the other, the division of a line in extreme and mean ratio. The first we may name as a measure of gold, the second we may name as a precious jewel."

In Greek mathematics, there was another remarkable number, the so called the *golden number* (or *golden ratio*) that is defined in geometry by dividing a straight line segment in such a way that the ratio of the total length l to the larger segment x is equal to the ratio of the larger to the smaller segment. In other words, the golden ratio, $g = (l/x)$ is determined by the equation

$$\frac{l}{x} = \frac{x}{l - x} \tag{1.2.10}$$

or, equivalently,

$$g^2 - g - 1 = 0. \tag{1.2.11}$$

The positive solution of quadratic equation (1.2.11)

$$g = \frac{l}{x} = \frac{1}{2}\left(\sqrt{5} + 1\right) = 1.618. \tag{1.2.12}$$

The inverse ratio of g is

$$\frac{1}{g} = \frac{x}{l} = \frac{1}{2}\left(\sqrt{5} - 1\right) = 0.618 \tag{1.2.13}$$

so that $\frac{1}{g} = g - 1$.

In geometry, the Pythagoreans developed the theory of space filling figures, whatever the motivation for their work, the Pythagoreans evidently considered the geometrical figures to be very important for space filling. For example, one of the diagrams (see Figure 1.6) shows six equal equilateral triangles filling space around their central point. But five such equilateral triangles can similarly be fitted together to generate a bell-tent-shaped figure around a central vertex so that their bases form a regular pentagon.

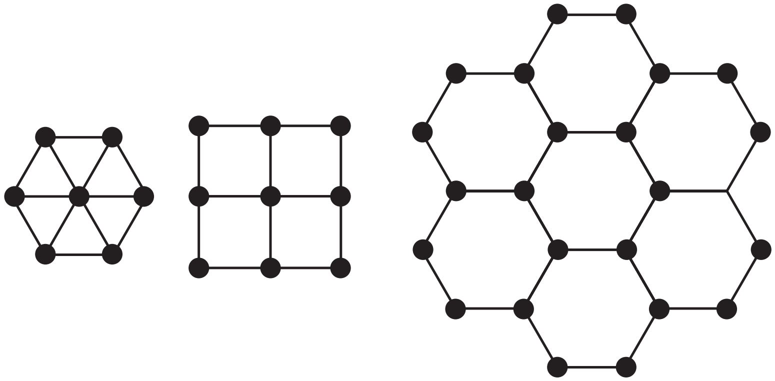

Fig. 1.6 Pythagorean or Six Equilateral Triangles filling space around the center.

Such a figure becomes a solid figure with the vertex of a regular *icosahedron*. This process can be repeated by surrounding each vertex of the original equilateral triangles with five triangles. Exactly twenty equilateral triangles are required to generate the beautiful solid figure of the icosahedron of twelve vertices and twenty faces. It is remarkable that in solid geometry there are exactly five such regular figures, known as *regular polyhedra* (or *Platonic solids*), and that in the plane there is a very limited number of regular space-filling geometric figures. The first three simplest regular polyhedra including *tetrahedron*, *cube* and *octahedron* were found by Egyptian mathematicians. Pythagorean discovered the remaining two — the icosahedron, and the *dodecahedron* with twenty vertices and twelve faces.

It is important to point out that a study of the properties of the regular pentagon led to the discovery of the golden ratio, the ratio in this case being that of the diagonal of the pentagon to its side. In Figure 1.7, the diagonal AC of the pentagon divides the diagonal BE into two unequal segments BP and PE such that the ratio of the smaller segment to the larger is equal to the ratio of the larger segment to the whole diagonal. In fact, any diagonal of the regular pentagon divides any other interesting diagonal in this way. Such division was known to the Greek mathematicians as "division of a line in mean and extreme ratio". We have already stated that this ratio is the golden ratio $g = (BE/PE)$, where $BE = \ell$ and $PE = x$ so that the algebraic formulation is $(\ell - x)/x = (x/\ell)$. This leads to the quadratic equation (1.2.11) in the golden ratio g.

THE LEGACY OF LEONHARD EULER - A Tricentennial Tribute

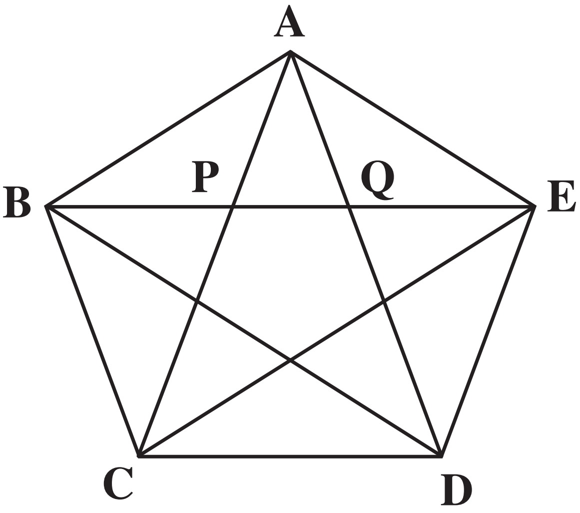

Fig. 1.7 A Regular Pentagon $ABCDE$.

Some of the angles associated with Figure 1.7 follow from the construction of the triangle ACD with angles $\angle ACD = \angle ADC = 72°$ and $\angle CAD = 36°$. It is then a simple matter to construct the complete pentagon so that $\angle ABC = 108°$, $\angle BAC = 36° = \angle BCA$, and hence, all angles of the pentagon are known.

It was Euclid of Alexandria (365-300 B.C.) made the first systematic development of Euclidean geometry in his famous treatise, *The Elements* in 13 volumes. These volumes represented a standard reference of geometry and number theory and a great model for the first axiomatic method in mathematics. He first proved that the number of primes is infinite which is one of the fundamental results in mathematics. However, the first completely rigorous axiomatic method of mathematics from a modern point of view was given by David Hilbert in his *Principle of Geometry* that was published in 1899. The Greek mathematicians also advanced other areas of mathematics and astronomy. Archimedes (287-212 B.C.) of Syracuse also made many other major contributions to mathematics and mathematical physics. He determined the center of mass of bodies and simple surfaces and derived the formula for the workings of levers and equilibrium of floating bodies. His major work for finding areas and volumes marked the birth of calculus. Archimedes was probably the last great mathematical scientists of ancient times. Another Greek mathematician, Claudius Ptolemy (85-169 A.D.) of Alexandria became famous for his major contributions to plane and spherical trigonometry and astronomy. Diophantus of Alexandria worked on theory of equations and earned the title of Father of Algebra. During the Middle Ages, the greatest discoveries in India were natural numbers including zero and the decimal number system. According to P. S. Laplace (1749-1827): "It is India that gave us the ingenious method of expressing all numbers by ten symbols, each symbol receiving a value of position, as well as an absolute value. We shall appreciate the grandeur of the achievement when we remember that it escaped the genius of Archimedes and Apollonius."

1.3 The Major Impact of the European Renaissance on Mathematics and Science

During the middle ages, the Italian mathematician Leonardo of Pisa (Fibonacci (1170-1250)) published his major book *Liber Abaci* in 1202, an influential book which introduced the Hindu-Arabic number system to Western Europe. The *European Renaissance*, from the 1400 to the 1600s produced many great advances in physics, astronomy, pure and applied mathematics. Michael Stifel (1487-1567), Nicolò Tartaglia (1506-1557), Girolamo Cardano (1501-1576), and Francois Viète (1540-1603) made major contributions to algebra, trigonometry and quadratic and cubic equations. Viète introduced the use of letters to stand for unknown quantities. Nicolaus Copernicus (1473-1543), the great astronomer who boldly rejected the fourteen-hundred year old Ptolemy's mathematical theory of astronomy with the Earth at the center of the universe and discovered the revolutionary modern heliocentric picture of the universe with the Sun at the center and made contributions to mathematics through his great work in astronomy with the publishing of *De revolutionibus orbium coelestium* in 1543.

Thoroughly convinced by the beauty and harmony of the Copernicus heliocentric system that the planets revolve in orbits about the sun at the center of the Universe, a great German mathematical scientist and astronomer, Johann Kepler used his brilliant imagination and amazing perseverance to modernize the Copernicus model in mathematical astronomy. As a research assistant to the famous Danish-Swedish astronomer, Tycho Brahe (1546-1601), Kepler had the rare opportunity to utilize Brahe's precise and extensive observational data. Based on these observational data, Kepler first discovered his three famous laws of planetary motion, the first two founded in 1609 and the third one ten years later in 1619. Kepler's laws of planetary motion are considered as major landmarks in the history of mathematical science and astronomy, for in the effort to justify them, Newton was led to discover modern celestial mechanics during 1660-1666. In his celebrated work of 1619, *Harmony of the World*, Kepler expressed his great satisfaction with the following statement in the preface:

"I am writing a book for my contemporaries or — it does not matter — for posterity. It may be that my book will wait for a hundred years for a reader. Has not God united for 6000 years for an observer?"

The 1600s brought many major discoveries in mathematics and astronomy. Two British mathematicians, John Napier (1550-1617) and Henry Briggs (1556-1631), first Savilian Professor of Geometry at the University of Oxford, invented logarithms to the base of 10. Logarithms to the base of 10 are usually known as Briggian logarithms, through the advantage of using this base appears to have occurred independently to Napier and Briggs. Napier published his book *Mirifici logarithmorum canonis descriptions*, in which logarithms are introduced in great detail. On the other hand, two Englishmen, Thomas Harriot (1560-1621), and William Oughtred (1557-1660) developed new methods for classical algebra. Galileo Galilei (1564-1642) an Italian astronomer and physicist and Johann Kepler, a German mathematician and astronomer tremendously expanded knowledge of mathematics and physics through their studies of astronomy, physics and mathematics. Galileo discovered the famous law of falling bodies which marked the beginning of modern experimental physics. He suggested that all bodies are attracted to the Earth by the constant gravitational acceleration regardless of their weights. His famous experiment dealt with dropping two unequal weights from the top of the Leaning Tower of Pisa. This became controversial because it contradicted Aristotle's (384-322 B.C.) old views that heavy bodies fell faster than lighter ones. It is also important to mention Galileo's work on the curve *cycloid* in 1630 and his suggestion that arches of bridges should be built in the shape of cycloid. The quadrature (or finding an area) of the cycloid has been calculated in 1630 by Evangelista Torricelli (1608-1647), a student of Galileo. About the same time, Pascal proved many new theorems about properties of cycloid and calculated the area of the segment of cycloid. This was followed by another remarkable discovery of a great Dutch mathematical scientist, Christian Huygens (1629-1695) in 1658 that was concerned with the solution of the problem of the tantochronous motion. Indeed, the cycloid is a true tantochrone that is, if a particle is allowed to slide from rest down a cycloid, it takes exactly the same time to reach the bottom, no matter where it starts from. Huygens also made another discovery that a pendulum bob swinging along a cycloid curve takes exactly the same time to make a complete oscillation whether it swings through a small or large arc. He made many new and sensational discoveries in physics and astronomy including his strong support for the Copernicus heliocentric model of the universe. In 1609, he build a telescope that has opened new worlds in astronomy, and has become an indispensable instrument for centuries for astronomy.

Galileo discovered the laws of pendulum and was credited for his most remarkable discovery of four bright satellites of the planet Jupiter in 1610. In the same year, he observed some peculiar form of the planet Saturn. His historic achievements in astronomy dealt with the discovery of many more and more powerful telescopes that were sold in Europe. This instrument has made it possible to study, observe and photograph many heavenly bodies which were formerly unknown. His name and fame as the greatest experimental scientist of his time attracted many scholars from all parts of Europe. Christian Huygens also built a powerful telescope which made possible his new discovery of satellites and the rings of Saturn. He was the first one who used a pendulum to regulate a clock and then applied the basic principles of pendulum in building astronomical clocks. In addition, he investigated the wave theory of light and discovered the polarization of light.

Galileo is universally considered as the founder of methodology of modern science. His radical departure from the Greeks, the medieval and contemporary scientists led him to establish the fact that matter as well as motion were only the first step to a new approach to nature. In 1632, he published his beautifully written masterpiece, *A Dialogue on the Two Principal Systems of the World* in which he gave a critical evaluation of the comparative merits of the old and new theories of motions of the celestial bodies. He spent considerable amount of time in writing on force and motion. In particular, his firm helicentric views of the universe was in severe disagreement with religion doctrines of the Inquisition. In 1638, he published his other greatest classic, *Dialogues on the Two New Sciences* in which he founded the modern science of mechanics. It contained his life's work on motion, acceleration, and gravity and provided a sound basis for the three laws of motion formulated by Sir Isaac Newton (1642-1727) in 1687.

Without being too precise, mechanics is simply the study motion of material bodies (or particles) that can be described by mathematical models. In mechanics, a body (particle) is supposed to be subject to certain forces, which affect its motion according to certain laws. Expressed in the language of mathematics, these laws usually take the form of differential equations, that is, they connect the position, velocity, momentum, acceleration of the body at a particular instant of time. They do not primarily describe the whole motion, but merely the laws governing it. It is the motion as a whole which has to be derived from the law. In other words, this is a problem of solving differential equations with time as the independent variable, and there are one or more dependent variables which determine the position of the body.

Galileo's two greatest classics are not only two profound books of all time, but they are clear, direct, truly powerful and fascinating in the history of science and philosophy. In general, his scientific philosophy and scientific method were in agreement with those of Descartes, Huygens, Newton and others. His new methodology of science led him to believe in the total reformulation that not only imparted expected and unprecedented power to science, but bound it indissolubly to mathematics. It was Galileo who remarkably discovered the more radical, more effective and more practical methods for modern science. He demonstrated the profound effectiveness of his approach to science through his own work. It is a delight to quote a philosopher, Thomas Hobbes (1588-1678) who said of Galileo: "He has been the first to open to us the door to the whole physics." Galileo himself was convinced that nature is simple, orderly, and mathematically designed which can be documented by his own famous 1610 quotation: "Philosophy [nature] is written in that great book which ever lies before our eyes — I mean the universe — but we cannot understand it if we do not first ... labyrinth."

Both Galileo and Newton strongly emphasized that mathematical principles are quantitative principles which played a vital role in providing the correct physical explanation of natural phenomena. They also believed that experiments are needed to establish basic laws of science. In the preface to his *Principia*, Newton expressed his firm views on the intimate relationship between the mathematical principles (or laws) and the natural phenomena as follows:

"Since the ancients (as we are told by Pappus) esteemed the science of mechanics of greatest importance in the investigation of natural things, and the moderns, rejecting substantial forms and occult qualities, have endeavored to subject the phenomena of nature to the laws of mathematics, I have in this treatise cultivated mathematics as far as it relates to philosophy [science] ... and therefore I offer this work as the mathematical principles of philosophy, for the whole burden in philosophy seems to consist in this — from the phenomena of motions to investigate the forces of nature, and then from these forces to demonstrate the other phenomena....”

Finally, we close this section by adding the most tragic event of Galileo’s life. In 1633, after a long and painful trial by a tribunal of the Inquisition because of his heliocentric view of the universe contrary to church teachings, he was sentenced to house imprisonment for the rest of his life. He remained a prisoner in Florence until his death in 1642.

During the Renaissance period, two great French mathematicians, René Descartes (1596-1650) and Pierre de Fermat (1601-1665) created a new branch of mathematics which is now known as *analytic geometry*. By the 1630s, both men discovered the basic idea of using algebraic equations to represent curves and surfaces and investigated their fundamental properties. Descartes’ major objective was to unify the hitherto largely two separate disciplines of algebra and geometry, in particular to use the algebraic method to solve geometrical construction problems. His great mathematical work dealing with applications of algebra to geometry was *La Géométrie*.

On the other hand, based on the work of Apollonius on conic sections, Fermat discovered the fundamental principle of geometry, which he expressed thus: “Whenever in a final equation two unknown quantities are found, we have a locus, the extremity of one of these describing a line, straight or curved.” This profound statement was written at least one year before the publication of Descartes’ *La Géométrie*. Fermat formulated his major ideas further in his short book entitled *Ad locus planos et solidos isagoge* (*Introduction to Loci Consisting of Straight Lines and Curves of the Second Degree*) which was published in 1679 – almost fourteen years after his death: That is why Descartes is widely recognized as the sole creator of coordinate geometry. However, it clearly follows from Fermat’s above quotation that his approach is undoubtedly more simple, direct and more systematic than that of Descartes. In the eighteenth century, the view that the algebraic approach to geometry was more than a mathematical tool. Algebra itself is a fundamental method of introducing and investigating curves and surfaces in general. All these simply mean that the analytic geometry paved the way for a complete unification of algebra and geometry from a modern point of view.

Based on the great work of the classical masters, Apollonius and Diophantus on geometry, in general and conic sections, in particular, Girard

Desargues (1591-1661) created a totally new branch of geometry in 1639 which is now know as the *Projective Geometry*. It deals with the study of the descriptive properties of geometrical figures. In other words, it is basically concerned with those geometrical properties which are unchanged by the operation of section and projection. The basic metrical properties in geometry which include distance, areas, angles, congruence, and similarity are not considered in projective geometry. For example, the Pythagorean Theorem for a right angled triangle ($c^2 = a^2 + b^2$), the area of a triangle of base a and height h $\left(\triangle = \frac{1}{2}ah\right)$ and the sum of the three angles of a triangle ABC $(A + B + C = \pi)$ are famous metrical theorems. Projective geometry has grown into a vast and beautiful branch of geometry through the major works of great French mathematicians including Desargues, Blaise Pascal (1623-1662), Gaspard Monge (1746-1818) and Jean Victor Poncelet (1788-1867). Like several other branches of geometry, it has become a new source of mathematical knowledge for the study of descriptive geometry.

The observed symmetry between points and lines in a projective plane leads to the so-called *principle of duality* which is one of the most elegant properties of the projective geometry. This basic principle states that, in a projective plane, every theorem (or proposition) remains true when the words *point* and *line* are consistently interchanged. Thus, given a theorem and its proof, we *can* immediately formulate the dual theorem whose proof can be written down mechanically by the use of the duality principle in every step of the proof of the original theorem.

Desargues not only introduced many ideas, notably the point and the line at infinity and gave elegant proofs of many new theorems. Above all, he first discovered the concepts of section, projection and cross-ratio of four points which were used to give a new method of proof. He then developed a unified approach to several types of conic sections through projections and sections. It may be appropriate to give some examples of basic theorems in projective geometry. One such example is the Desargues' famous two-triangle theorem which is illustrated in Figure 1.8 with a vortex O and the triangle $A^{'}B^{'}C^{'}$ is obtained from the triangle ABC by projection and section from the vertex O. Desargues' theorem then states that the three pairs of the corresponding sides AB and $A^{'}B^{'}$, BC and $B^{'}C^{'}$, and AC and $A^{'}C^{'}$ (or their extensions) of two triangles perspective from a point meet in three colinear points L, M, N as shown in Figure 1.8. Conversely, if the three pairs of corresponding sides of the two triangles meet in three points that lie on one straight line, then the line joining corresponding vertices meet in one point. In other words, making reference to Figure 1.8,

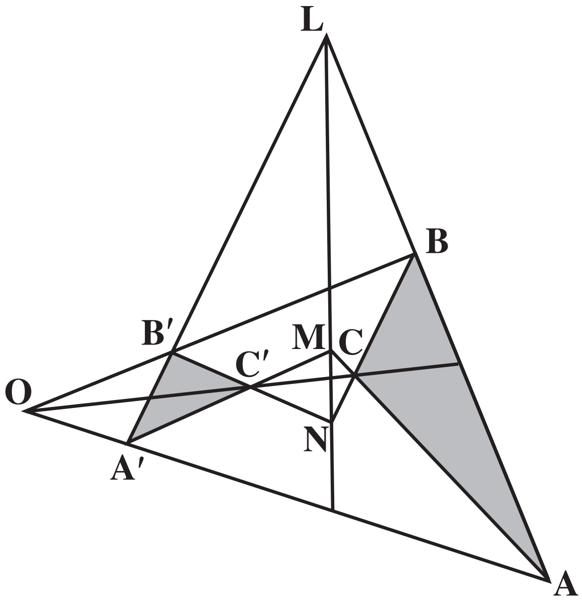

Fig. 1.8 The Desargues two triangles.

the converse theorem asserts that since $AA^{'}$, $BB^{'}$, and $CC^{'}$ intersect at a point O, the sides AB and $A^{'}B^{'}$ intersect at a point L, AC and $A^{'}C^{'}$ meet in a point M and BC and $B^{'}C^{'}$ meet at N; then L, M, and N lie on a straight line. It is important to note that both theorems are true whether the triangles ABC and $A^{'}B^{'}C^{'}$ lie on the same or different planes. Desargues gave an elegant proof of his theorem and its converse for both two- and three-dimensional cases.

The second major contributor to projective geometry was Pascal. In 1640, at the age of sixteen, Pascal gave a pleasant surprise to the world by publishing a short book entitled *Essai pour les coniques* in which he described his celebrated theorem of the *hexagrammum mysticum (Mystic Hexagram)*. It is universely known as the *Pascal Theorem* which is illustrated in Figure 1.9, and it states that the three pairs of opposite sides of a hexagon inscribed in a conic meet in three collinear points. In other words, making reference to Figure 1.9, if BA and DE intersect at L, CD and AF intersect at M, and BC and FE intersect at N, then L, M, N lie a straight line. Conversely, if a hexagon is such that the points of intersection of its three pairs of opposite sides lie on a straight line, then the vertices of the

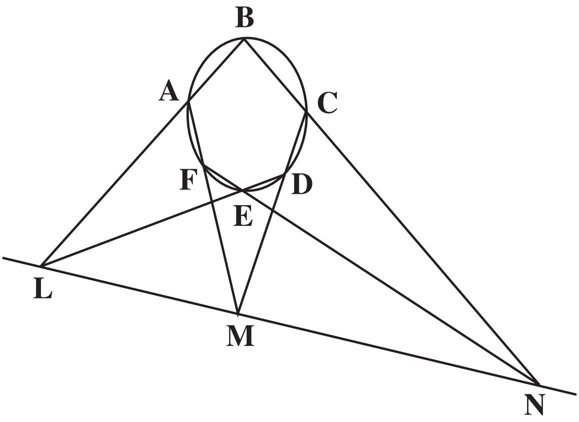

Fig. 1.9 The Pascal hexagon in a conic.

hexagon lie on a conic. However, Pascal did not give an explicit proof of his theorem and its converse. He simply stated that his theorem is true for a circle and it must also be true for a conic by the method of projection.

Desargues regarded Pascal's 1640 essay was so brilliant that he could not believe it was written by such a young man. He encouraged Pascal to do more research on projective geometry in order to develop the method of projection and section further. At the advice of Desargues, Pascal began working on conics and used projective methods, that is, projection and section. He admired Desargues' work and acknowledged his debt to Desargues by saying : "I should like to say that I owe the little that I have found on this subject to his writings."

In addition to his contribution to projective geometry, Pascal made major contributions to the mathematical theory of probability. In 1654, a French man, the Chevalier de Méré, suggested some problems associated with games of chance. During that time, Pascal had some correspondence with Pierre de Fermat dealing with these problems of games of chance and gambling in general. Thus, the first research collaboration of Pascal and Fermat on problems of games and chance led to mark the birth of the mathematical theory of probability which is now widely used in mathematical statistics. Based on some results of Fermat and Pascal, Christian Huygens wrote the first treatise on probability in 1657. About the same time, Pascal wrote a treatise entitled *Traité du triangle arithmétique* which included the coefficients of the binomial expansion

$$(a+b)^n = \sum_{r=0}^{n} \tbinom{n}{r} \quad a^{n-r} \quad b^r, \tag{1.3.1}$$

so that, when $a = b = 1$,

$$\sum_{r=0}^{n} \tbinom{n}{r} = 2^n, \tag{1.3.2}$$

the number of *combinations* of n objects taken r at a time is,

$$^nC_r = \tbinom{n}{r} = \frac{n!}{(n-r)!\; r!}, \tag{1.3.3}$$

the Pascal recurrence relation

$$^nC_r +^n C_{r-1} = \ ^{n+1}C_r, \tag{1.3.4}$$

and the coefficients of the binomial formulas organized into famous *Pascal's triangle*. Pascal made an extensive study of the properties of his triangle, in the course of which he discovered the *principle of mathematical induction*. This principle, which states the validity of the mathematical argument by recurrence, is now considered as one of the fundamental axioms of modern mathematics. Many proofs in mathematics are based on the famous *principle of induction*. Pascal became a renowned scientist in Europe for his fundamental works in geometry, hydrostatistics and probability theory. He also invented a new calculating machine which is still preserved in a French museum.

After a century of slow progress, the revival of the projective geometry received considerable attention by Gaspard Monge (1746-1818) and his school at the École Polytechnique. Monge's extensive work in descriptive geometry, ordinary and partial differential equations won the remarkable admiration from mathematical scientists of the world. His greatest student was Poncelet who published his famous *Treatise on the Projective Properties of Figures* in 1822 which he subsequently expanded and revised this treatise and later published in two volumes entitled *Applications d'analyse et de géométrie* (1862-1864). All these published works were his major contributions to projective geometry and to the creation of modern projective geometry. He was the first mathematician to recognize fully that projective geometry was a new branch of geometry with methods and results of its own. He formulated the general problem of seeking all properties of geometrical figures which were common to all sections of any projection of a figure, that is, remained unchanged by projection and section. His work was essentially based on three major ideas: homologous figure, principle of continuity, and pole and polar with respect to a conic. Two figures are called *homologous* if one can be derived from the other by one projection and a section. In his 1822 *Traité*, Poncelet phrased the principle of continuity as: “If one figure is derived from another by a continuous change and the latter is as general as the former, then any property of the first figure can be asserted at once for the second figure.” He advanced the principle as an absolute truth and used it in his *Traité* to prove many theorems, and then generalized the principle to make assertions about imaginary figures. The concept of pole and polar with respect to a conic was the third major idea in Poncelet’s work. He gave a general formulation of the transformation form pole to polar and conversely. His major objective in studying polar reciprocation with respect to a conic was to establish the principle of duality in projective geometry. By virtue of this principle, lines can be as fundamental as points in the development of plane projective geometry. Like others, Poncelet recognized that theorems dealing with figures lying in one plane when interchanged the word point by line and vice versa not only made sense but proved to be true in general. It is fair to say that all major contributors to projective geometry made the significant efforts to elevate the subject to hew heights of rigor, clarity, elegance and generality.

The Renaissance mathematical scientists including Copernicus, Brahe, Kepler, Galileo, Pascal, Huygens, Descartes, Newton and Leibniz spoke repeatedly of the cohesiveness and harmony that God imparted to the Universe through His *mathematical laws* and *design*. These men discovered mathematical knowledge that would reveal the grandeur and glory of God’s creation. Once Galileo said: “Nor does God less admirably discover Himself to us in Nature’s action than in the Scriptures’ sacred dictions.” Towards the end of the Renaissance period, many European scientists became very active in the formation of scientific societies or research institutes in order to stimulate more scientific research and to increase communication among mathematical scientists. Although the Italian academies and professional societies were founded in the early seventeenth century with Galileo and his students as members, but, unfortunately, they were disbanded after a while. For example, in France, several mathematical scientists including Desargues, Descartes, Fermat and Pascal met informally under the leadership of Marin Mersenne (1588-1648) to organize the Academie Royale des Sciences in 1630s. In England, John Wallis began in 1645 to hold meetings in Greshan College, London in order to establish a similar organization in England. This informal group was chartered by Charles II in 1662 and established the Royal Society of London for the promotion and dissemination of scientific knowledge. Its first president was a famous mathematician, Lord William Brouncker (1620-1684). *The Philosophical Transactions of the Royal Society* began its publication in 1665 and it was one of the first research journals to include mathematical and scientific papers. The French Academy of Sciences was founded by Colbert in 1666. The famous Lucasian Chair of Mathematics was established at the University of Cambridge in 1663 by Henry Lucas (1610-1667) who was a former student of Cambridge. The first professorship in mathematics was established at the University of Oxford in 1619. John Wallis became of professor of mathematics at Oxford in 1649 and held the Chair of mathematics until 1702. On the other hand, Gottfried Wihelm Leibniz (1646-1716) in Germany provided a major leadership role for some years to establish the Berlin Academy of Sciences in 1700 with Leibniz as its first President. In Russia, Peter the Great founded the Academy of Sciences at St. Petersburg in 1724. These academies and their scientific journals opened new outlets for mathematical and scientific communication first in Europe and then in other nations of the world. They not only promoted new scientific research, but also supported scientists for the cultivation of mathematics and sciences and for making mathematics and science more useful for the society. These professional organizations played the major role in advanced study and research, and in dissemination of scientific and mathematical knowledge throughout the world.

1.4 The Discovery of Calculus by Newton and Leibniz

Historically, Sir Isaac Newton and Gottfried Wihelm Leibniz independently discovered the calculus in the seventeenth century. In recognition of this remarkable discovery. John Von Neumann's (1903-1957) thought seems to worth quoting. "... the calculus was the first achievement of modern mathematics and it is difficult to overestimate its importance. I think it defines more equivocally than anything else the inception of modern mathematics, and the system of mathematical analysis, which is its logical development, still constitute the greatest technical advance in exact thinking."

Both Newton and Leibniz recognized that calculus can be regarded as the branch of mathematical study which treats change and the rate of change. They also observed the close connection between algebra and geometry, epitomized by the fact that every equation has a graph and every graph an equation.

By 1664, the young Newton became familiar with all mathematical ideas and results of his predecessors and was fully ready to discover his own. In his new analytical methods, Newton remarkably combined the ideas, results and methods of three largely separate branches of mathematics: coordinate geometry, calculus and infinite series (or more precisely, the representation of functions by infinite series). During 1664-1666, Newton developed all the basic ideas and methods in his first version dealing with the fluxional calculus. In this work, he treated variables as moving quantities changing with time and introduced the concept of velocity and acceleration at any instant of time. He then considered exposition of his ideas and results in the book *Methodus Fluxionum et Serieum Infinitarum* (*The Method of Fluxions and Infinite Series*) which was not published until 1736, after his death. In his book, Newton treated variables as flowing quantities generated with time by the continuous motion of points, lines and planes, rather than as static infinitesimal quantities as in his first version of the calculus. He defined a variable quantity x or y as the *fluent*, and its rate of change with respect to time as the *fluxion* which was denoted by $\dot{x}$ and $\dot{y}$ (the Newtonian dot notation) which is now known as the *derivative* or the *velocity.* Subsequently, he stated more clearly the fundamental problem of calculus by introducing any two variables rather than the time as the independent variable. For example, $y = x^n$ so that, in modern notation, $(dy/dx) = n\,x^{n-1}$. One of the outstanding problems of the seventeenth century was that of finding the tangent to a curve at an arbitrary point. It was solved by Newton's teacher at Cambridge University, Isaac Barrow (1630-1677). Newton developed the idea of the rate of change from the analytic point of view. He also demonstrated his ideas by examples of finding tangents to well-known plane curves including cycloid and spirial. He gave another example of a plane curve with its algebraic equation in the form

$$x^3 - ax^2 + axy - y^3 = 0, \tag{1.4.1}$$

to derive the fluxional equation

$$3x^2\dot{x} - 2a\,x\,\dot{x} + a(\dot{x}y + x\dot{y}) - 3y^2\dot{y} = 0. \tag{1.4.2}$$

This gives the slope (or gradient) of the tangent to the curve at any point (x, y) so that

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{(3x^2 - 2ax + ay)}{(3y^2 - ax)}. \tag{1.4.3}$$

In his book, Newton not only developed a general method for finding the instantaneous rate of change of one variable with respect to another, but proved that area can be found by reversing the process of finding a rate of change. For example, if the curve is $y = amx^{m-1}$, the area under the curve is $z = ax^m$. In modern notation, this result can be written as

$$\text{Area} = z = \int_0^x amx^{m-1}dx = ax^m. \tag{1.4.4}$$

He applied the method to calculate the area of many plane curves. Since areas can be computed by the summation of infinitesimal areas, thus the summations (or more precisely, the limits of sums) can be obtained by reversing the process of differentiation. This is now known as the *fundamental theorem of integral calculus*. He also derived the correct formula for the curvature of a given curve, namely,

$$\kappa = \frac{\ddot{y}}{(1+\dot{y}^2)^{3/2}}. \tag{1.4.5}$$

Newton obtained the area under the curve $(dz/dx) = (1 + x^2)^{-1}$ in terms of Gregory's infinite series as

$$z = \tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots\,. \tag{1.4.6}$$

Newton's major idea of infinite series dealt with the discovery of the general binomial theorem as well as the binomial infinite series which was then applied to solve the problems of calculus. The study of infinite series led to another very important method in mathematics which deals with the solution of problems by means of *successive approximations*. This means that we first find an approximate solution of a problem, then based on this, we look for a still better solution, or a second approximation; and so on, each time getting a little better result to the exact solution. This process can be continued to find the best approximate solution. For example, if $f(x)$ is a continuous function, naturally $f(x + h)$ is approximately equal to $f(x)$ if h is small. This implies that $f(x)$ is a first approximation to $f(x + h)$, and we may write this as

$$f(x + h) = f(x) + \cdots\,, \tag{1.4.7}$$

where $+\cdots$ means that there is still something lacking to the exact solution. To obtain a second linear approximation, we use the definition of the first derivative, that is, $f'(x)$ is the limit of

$$\frac{f(x+h) - f(x)}{h} \quad \text{as} \quad h \to 0. \tag{1.4.8}$$

Consequently, $f(x+h)-f(x)$ is approximately equal to $h\, f'(x)$ so the next linear approximation (differentiation as *linear approximation*) is

$$f(x + h) = f(x) + h\, f'(x) + \cdots\,, \tag{1.4.9}$$

where $+\cdots$ has the same meaning as before. Continuing the process, we obtain the third approximation formula

$$f(x+h) = f(x) + h\, f'(x) + \frac{h^2}{2!} f''(x) + \cdots . \tag{1.4.10}$$

Thus, it is possible to continue this process so as to obtain still further closer approximations. Eventually, this leads to the celebrated formula known as *Taylor's series expansion* of $f(x+h)$ in the form

$$f(x+h) = f(x) + h\, f'(x) + \frac{h^2}{2!} f''(x) + \cdots + \frac{h^n}{n!} f^{(n)}(x) + \cdots , \tag{1.4.11}$$

where $f^{(n)}(x)$ is the nth derivative of $f(x)$. Brook Taylor (1685-1731), a British mathematician knew this formula in 1712 without any rigorous proof, but his name has become inseparably associated with the formula. However, the Taylor series was known to another British mathematician, James Gregory (1638-1675) in 1670. Indeed the Taylor formula (1.4.11) originated from the Gregory–Newton interpolation formula for the calculus of finite differences involved in simple and elementary functions, it was apparently not discovered by Newton who, of course, knew some special cases of it. Putting $x = 0$ in (1.4.11) and replacing h by x leads to the famous Maclaurin series

$$f(x) = f(0) + x\, f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0) + \cdots . \tag{1.4.12}$$

This was deduced by Colin Maclaurin (1698-1746) who gave this special case in his *Treatise of Fluxions* published in 1742 and stated that it was a special case of Taylor's result (1.4.11). In his Treatise, Maclaurin made a serious attempt to establish the Newton's calculus more rigorous. It was a commendable effort but not successful due to the lack of convergence. Maclaurin succeeded James Gregory as professor of mathematics at the University of Edinburgh. On the other hand, Gregory discovered another simple but important infinite series for $y = \tan^{-1} x$ as

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots . \tag{1.4.13}$$

This is universally known as *Gregory's series*. The formula for π known as Gregory's formula is the special case of (1.4.13) by taking $x = 1$ so that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots . \tag{1.4.14}$$

This series was also discovered by Leibniz in 1674. Many mathematicians including Newton, Leibniz, Gregory, Maclaurin and Euler were interested in infinite series. One of the major uses of infinite series beyond their service in differentiation and integration was to find the series expansion of functions such as trigonometric functions, exponential and logarithmic functions, and π and e.

In his student days, Newton had made an extensive study of Descartes *La Geometrie* which prepared him well to pursue advanced study and research in geometry. In the late 1660s, he embarked on an extensive research in algebraic equations and the method of coordinate geometry. Using the method of coordinate transformations, he described fairly general geometrical curves with many examples of cubic curves. He proved that the general cubic equation containing ten terms can be expressed in the simpler form

$$axy^2 + bx^3 + cx^2 + dy + ex + f = 0. \tag{1.4.15}$$

In about 1690, Newton revised and expanded his earlier work on geometry concerned with a large number of general theorems about cubic curves and their asymptotes, and the associated conic sections. According to some mathematical historians, it seemed that Newton had a comprehensive plan to write three-volume treatise on the *Geometria*, but his work was never completed.

Sir Isaac Newton's discovery of the calculus and the fundamental mathematical and physical laws were published in his first book of *Philosophiae Naturalis Principia Mathematica* (*Mathematical Principles of Natural Philosophy*) which is considered one of the greatest single contribution ever published in the history of physical sciences. This celebrated volume, usually called *Principia* or *Principia Mathematica* was completed over three hundred years ago and communicated to the Royal Society in the Spring of 1686 and then published in 1687. In it Newton not only put forward a new theory of how bodies move in space and time, but also developed the complicated mathematics needed to analyze these motions. In addition, he also profoundly formulated the laws of motion and a law of universal gravitation according to which each body in the universe was attracted toward every other body by a force that was stronger when bodies are more massive and close to each other. It was exactly the same force that caused objects to fall to the ground. According to his law, gravity causes the Moon to move in an elliptic orbit around the Earth and the planets to follow elliptical paths around the Sun. It was the first book by Newton to contain a unified system of scientific principles explaining what happens on the Earth and the Universe.

From the time of the publication of the Newton's *Principia* and throughout the eighteenth century, the Newtonian world-view was the remarkable influence on the British intellectual life, especially, in the fields of mathematics, physics, astronomy and natural philosophy. It may be appropriate to mention at least four British mathematical scientists including John Wallis who was Newton's contemporary and one of his closest friends, and Isaac Barrow – Newton's teacher and friend. Two other younger colleagues and friends of Newton were Edmond Halley (1656-1742) and Roger Cotes (1682-1716) who were closely involved in the editing and publication of Newton's greatest work. Wallis' research on algebra and his two major books entitled *Tractatus de sectionibus conicas and Arithmetica Infinitorum* have had tremendous influence on Newton's discovery of the general binomial theorem as well as the calculus. It was equally remarkable that Newton's teacher Barrow was fully responsible for providing adequate training and help so that Newton can become the great mathematical scientist of all time. Indeed, Barrow provided numerous and generous help to his young student and friend in many different ways. For example, Barrow became the first Lucasian Chair of Mathematics at Cambridge University in 1663. In 1669, he suddenly resigned his Lucasian Chair and encouraged the University to offer this prestigious Chair to young Newton as Barrow strongly believed that Newton was the most outstanding mathematical scientist for this position.

In his independent discovery of the calculus, Leibniz began with a fairly general approach to infinitely small increments in x and y, where δx is used to indicate the difference of two infinitely close values of x, and δy to indicate the difference of two infinitely close values of y. The limiting value of the ratio $\left(\frac{\delta y}{\delta x}\right)$ as δx tends to zero is written as $\frac{dy}{dx}$ or sometimes as $D_x y$ or Dy, and is called the *derivative* of y at a point x of the curve $y = f(x)$. Geometrically, $\frac{dy}{dx} = \tan\theta$ represents the slope of the tangent to the curve $y = f(x)$ at x, where the tangent at the point x makes an angle θ with the positive direction of the x-axis. If $y = f(x)$, then its derivative of y at x is often written as $f'(x)$. Similarly, the derivative of a derivative, that is, $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is written as $\frac{d^2y}{dx^2}$ or $f''(x)$ is called the *second derivative.* (The prime notation or the symbol D is due to Leibniz.) The process can be continued to give the nth derivative written as $D^n y = \frac{d^n y}{dx^n}$ or $f^{(n)}(x)$. So, in his discovery of the calculus, Leibniz first introduced the idea of symbolic method and used the symbol $\frac{d^n y}{dx^n} = D^n y$ for the nth derivative, where n is a non-negative integer. L' Hospital (1661-1704), student of Johann Bernoulli (1667-1748), asked Leibniz about the possibility that n is a fraction, "What if $n = \frac{1}{2}$?". In 1665 Leibniz replied, "It will lead to a paradox." But he added prophetically, "From this apparent paradox, one day useful consequences will be drawn." Unlike Newton, Leibniz was more concerned with operational formulas to develop the ideas, methods and results of the calculus in the broad sense. For example, Leibniz proved the formulas for the derivative of a product or quotient of two (or more) functions. His formula for $D^n(uv)$, where u and v are functions of x, and a table of integrals provided the basic rules and formulas in calculus. Newton used his fundamental concept of the fluxion to solve the problems of area and volume. According to Newton, the fundamental theorem of calculus is a direct consequence of his definition of integration, that is, the fluent can be calculated from the fluxion. Indeed, he created the infinitesimal calculus and first formulated the ideas of fluxions and fluent as early as 1664-1666, and soon developed it into a general operational method. On the other hand, Leibniz first introduced the concept of integration as summation, so his definition does not imply the fundamental theorem of integral calculus which has to be proved. This idea led him to formulate the general problem of finding the area of the curve $y = f(x)$ between the portion of the x-axis from $x = a$ and $x = b$ and on the left and right by two straight line parallel to the y-axis. We divide the x-axis into n equal subintervals so that each of the subintervals along the x-axis is δx which is the base of every one of the small rectangular areas. The height of the average rectangle is represented by a perpendicular line drawn from a typical interior point of the interval δx to the curve. Its value is, of course, $f(x)$. The area of each such average rectangle is $f(x) \cdot \delta x$, and the sum $\sum f(x)\delta(x)$ of these many small elements of area is called the *definite integral* of the function $y = f(x)$ between the values of $x = a$ and $x = b$. In the notation of Leibniz, the limiting value of the sum representing the total area A is equal to the definite integral in the form

$$A = \int_a^b f(x)dx, \tag{1.4.16}$$

where in the above sum, δx is replaced by dx and the sum by the integral as $\delta x \to 0$. Although he was in possession of his fundamental ideas, methods and formulas from 1670 onwards, Leibniz discovered the differential and integral calculus, as we have it today, during 1675 and 1685. His first paper on the subject was published in Latin in the 1684 issue of the *Acta Eruditorum Lipsienium* which was the first monthly scientific journal in Germany founded by Leibniz in 1682. So, his discovery of calculus soon became widely known in Europe, largely due to voluminous work of the Bernoulli brothers, Jakob and Johann, published in the *Acta* in the 1690s. Following the notations and results of Leibniz, the first textbook on the differential calculus, *Analyse des infiniment petits* was published by L' Hospital in 1696.

It is clear from the discovery that the problem of the area is a typical problem of the integral calculus, but there are many other major problems such as the length of a given curve, the volume of a solid body of a given shape, and the area of its curved surface. A typical problem in mechanics is: How far will a particle moving with a given law of velocity go in a given time?

Apparently, the differential and integral calculus seemed to be quite different and independent subjects. Indeed, there is a very close relationship between them. We now look at the formula (1.4.16) of the area very closely. The area of a curve with one curved side $y = f(x)$ and the base at $x = 0$ and $x = x$ is

$$F(x) = \int_0^x f(x)dx, \tag{1.4.17}$$

where the upper limit of the integral is an arbitrary but fixed number, so that the area $F(x)$ will depend on x. The natural question is: What is the derivative of $F(x)$? According to the rule of derivative, we can write

$$\frac{F(x+h) - F(x)}{h} \tag{1.4.18}$$

and proceed to the limit as $h \to 0$. Obviously, $F(x+h) - F(x)$ is the area between the curved boundary, the x-axis and the vertical lines $y = x$ and $y = x + h$. If h is very small, an appropriate figure reveals that this area $h \cdot f(x)$ so that

$$\frac{d}{dx} F(x) = f(x), \tag{1.4.19}$$

that is, the derivative of the integral is equal to the value, at the upper limit of integration, of the integrand. Differentiation of the integral naturally leads to the original function $y = f(x)$. In other words, differentiation is the inverse process of integration. This is a very major discovery, because it is usually very much easier to do differentiation than integration.

In integral calculus, many problems involve the integration of $(1/x)$ and the integral of this function is $\log_e x = \ln x$. Thus, the number e arises naturally as the base of logarithms. More precisely,

$$\int_1^x \frac{1}{x}\, dx = \log_e x. \tag{1.4.20}$$

If $y = \log_e x$, then $x = e^y$ so that $\frac{dy}{dx} = \frac{1}{x}$. Sometime, formula (1.4.20) is used as the definition of a logarithm. If this is made the starting point, the exponential function appears as the inverse of the logarithm function and vice versa.

However, both Newton and Leibniz soon recognized some major logical difficulties associated with the concept of infinitely small quantities and the limiting value of the ratio of two such small quantities. It was the celebrated French mathematician, Jean d'Alembert (1717-1783) who also recognized the extraordinary power and usefulness of the calculus in finding the solution of real-world problems. At the same time, he realized the lack of rigor in the calculus, and made a serious attempt to revive the concept of *limit* in order to give the logical foundation of the subject. In his famous Encyclopedia article on *Différentiel* published in 1754, d'Alembert used his own limit concept to explain and justify the rules of differential calculus. He presented the familiar chord and tangent figure and states that: "The differentiation of equations consists simply in finding the limit of the ratio of the finite differences of two variables included in the equation". This was essentially no more than a reformulation of the ultimate ratios of his predecessors. In 1768, d'Alembert published a short exposition of his ideas entitled *Sur les principes métaphysiques du calcul infinitésimal.* This elegantly written article served as a mathematical model of clarity and logical proof. Its objective, to quote the closing sentence, was to "provide a sufficient introduction to the subject for those who merely wish to have a general, but correct, idea of its principles." It was the nineteenth century refinement of the fundamental idea of limit that eventually resolved the basic problems of the calculus. The mathematical foundations of calculus was then firmly established in the first part of the nineteenth century through the basic concepts of analysis such as function, continuity, limit, differentiability, integrability and convergence notably due to Augustin-Louis Cauchy (1789-1857) who was considered as one of the greatest mathematical scientists in terms of rigor, clarity, elegance and generality. He is also regarded as the father of the theory of functions of a complex variable and the theory of mathematical elasticity.

Although Newton first discovered calculus in 1664-1666, and communicated his ideas and results through manuscripts and letters to selected friends from 1666 onwards, however, he never published his manuscripts during 1664-1686. In his two letters addressed to Leibniz in 1676, Newton made no mention of his 1671 manuscript "*Treatise of the method of series and fluxions*" which contained algorithms and rules of differential calculus (similar to those of Leibniz) and their applications to problems of tangents and curvatures of plane curves. Ultimately, his work on calculus was first published in his *Principia* in 1687. In the years between the publication of the second edition of *Principia* in 1712 and the death of Leibniz four years later, there had been a bitter controversy between the supporters of Newton and those of Leibniz over the priority of the discovery of the calculus. As President of the Royal Society, Newton himself participated in the priority dispute and claimed that he discovered calculus before Leibniz. However, it was soon realized that Newton's notations, the terms *fluent* and *fluxion* were far inferior to Leibniz's elegant symbolism, the concepts of differentiation and integration. Subsequently, mathematicians began using Leibniz's notations, the term *integral* instead of *fluent*, and *derivative* instead of *fluxion*. So, the Newtonian terminology became almost obsolute in literature. In his 700-page long book on calculus published in 1797, S. F. Lacroix (1765-1843) expressed his views on calculus as: "... The school of Leibniz had a marked superiority over that of Newton, due perhaps more to the superiority of the former's methods than to the genius of his disciples, the Bernoulli's .... When Newton's writing were circulated on the Continent, one could see that he was in possession of the method of fluxions well before Leibniz had invented his differential calculus; but while it was possible for Newton's genius to deduce everything from his method that Leibniz could deduce from his own, the latter could be applied much more easily than the former." At any rate, Newton might have avoided the priority dispute of calculus with Leibniz if he had published his work immediately after its completion in 1666. We now close this section with a special tribute to both great men, Newton and Leibniz , for their independent discovery of the calculus which, indeed, was the greatest intellectual achievement in the history of mathematical sciences.

The greatest achievement of the seventeenth century mathematics was calculus which, next to number theory, algebra, analytical geometry, and projective geometry, is the greatest creation in all of mathematical sciences. In addition, the methodology of modern science, Newtonian mechanics, the universal law of gravitation, astronomy and celestial mechanics have been around for some decades, and a wide variety of specific problems have been solved by new and ingenious methods. All these marked the beginning of the golden age of modern and useful mathematics and science. Fully equipped with an enormous amount of knowledge and information, Leonhard Euler appeared as the universal genius at the center stage of eighteenth century mathematics and became ready to discover, unify and disseminate scientific and mathematical knowledge.