Chapter 1

Dominant Balance

Isaac Newton was born in Woolstorpe, Lincolnshire, England, in 1642.[1] An intense and lonely boy, he began early to keep notebooks of his ideas and calculations, a practice he continued all his life. He also had mechanical talents and while in school constructed many models in wood, including water clocks, sun dials, and a watermill. Early influences included Aristotle and Euclid, but also Descartes and Galileo. He graduated from Cambridge in 1665. While at Cambridge he developed the general binomial theorem and his work with this and other power series led to the formulation of differential and integral calculus (Turnbull [1951]); (Boyer [1959]), necessary tools for his investigation of planetary motion. Intensely secretive and easily hurt by criticism, he kept these results to himself, only revealing what he knew when he believed that someone else was close to discovering it and taking the credit away from him. His submission to the newly formed Royal Society of his discovery that white light was composed of all colors, which could be separated by a prism, led to a bitter dispute with Hooke, resolving him to further secrecy. Finally the astronomer Edmond Halley asked him in 1680 what path a comet would take moving in the gravitational field of the sun if the force of attraction varied inversely as the square of the distance. Newton gave the answer, an ellipse with the sun at one focus, and remarked that he had worked it out twenty years earlier. Halley convinced him to publish these results, and Newton then wrote the *Principia*, published in 1687. In this work appears for the first time the expression $e^x = \sum x^n/(1 \cdot 2 \cdot 3 \ldots n)$ and geometric methods of solving equations with small

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[1] Sources for biographical material for this and other chapters include Franceschetti [1999], Feingold [2004], Turnbull [1951] [1961], Bell [1965], Gleick [1992] [2003], Airy [1896] and Watson [1922], along with the MacTutor History of Mathematics Archive [2003].

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parameters, which he used to find approximate solutions to algebraic equations. These methods have been generalized and further developed by Kruskal [1963].

1.1 Introduction

Asymptotic analysis often involves the evaluation of mathematical expressions which contain a small parameter $0 < \epsilon \ll 1$. To simplify the solution of equations involving such expressions it is often possible to find two or more large terms which dominate the solution, other terms giving only small corrections to the value obtained by neglecting them entirely. In this case an iteration procedure can often be constructed which will give the solution to any degree of accuracy. The simplest such expressions are polynomials of the form

$$\sum_{p,q} C_{p,q}\epsilon^q x^p = 0, \tag{1.1}$$

with $C_{p,q}$ constants of order unity with respect to ϵ, and the sum over an index set for p, q. To find solutions of this equation using the idea of dominant balance we assume a particular ordering of a solution, i.e. take $x \sim \epsilon^\alpha$. The polynomial then becomes

$$\sum_{p,q} C_{p,q}\epsilon^{q+\alpha p} = 0. \tag{1.2}$$

Note that terms of the same order in ϵ in this equation are represented by a line $q + \alpha p = constant$. Each term in the polynomial can be represented as a point in the p, q plane, as shown in Fig. 1.1. All points above the line represent terms smaller than points on the line, and all points below the line represent terms larger than those on the line. A shift to a position with $q + \alpha p$ differing by an integer represents a difference in magnitude by one order of ϵ.

To find all possible combinations of dominant balance for a given polynomial, find all possible placements of a line so that it includes two or more terms of the polynomial, with all other points lying above the line. Each such placement represents a potential solution to the equation whereby the dominant balance of the solution is given by the points on the line, and all points above the line are associated with terms which are small corrections to this solution. Graphically this may be understood as bringing the line up from below until it makes contact with a point, and then rotating it

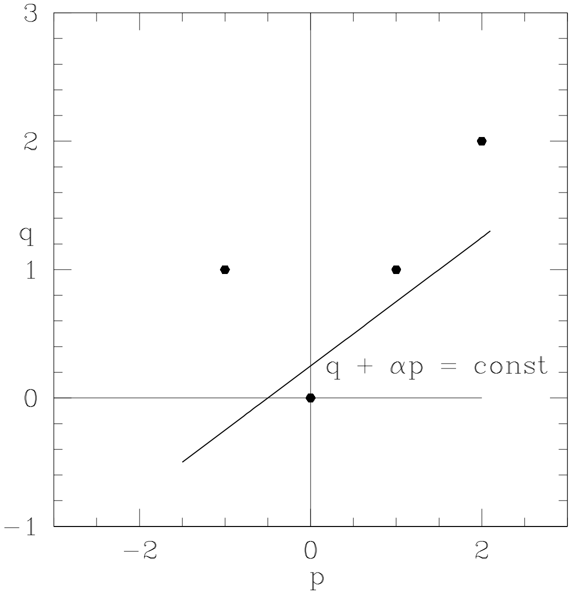


Fig. 1.1 Kruskal–Newton diagram.

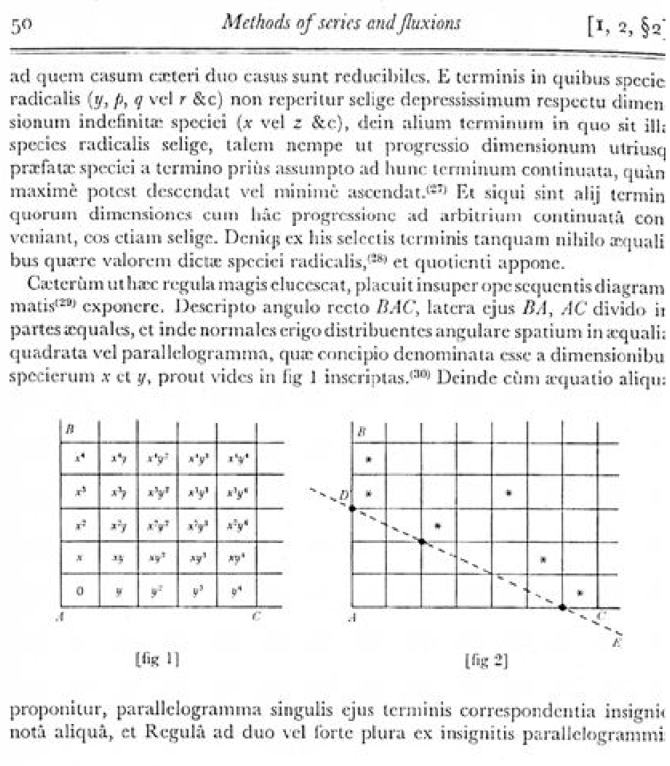
one way or the other until it makes contact with a second point. Such a plot is known as a Kruskal–Newton diagram, and is best understood with a few examples. Shown in Fig. 1.2 is a page from Newton's book on the method of fluxions, showing the first such diagram used by Newton to find the dominant terms in a problem illustrating the development of differential calculus. Figure 1 shows the placement for terms of the form x^q, y^p, arranged linearly in the variables p, q. The actual terms in the calculation are shown in Fig. 2, with the line DE passing through dominant terms, with higher order terms indicated by stars above the line.

1.2 Solutions Using Kruskal–Newton Diagrams

1.2.1 *Third order*

Consider the polynomial

$$1 - x - \epsilon x^2 - 2\epsilon^3 x^3 = 0, \tag{1.3}$$



50 *Methods of series and fluxions* [1, 2, §2

ad quem casum cæteri duo casus sunt reducibiles. E terminis in quibus specie
radicalis (*y*, *p*, *q* vel *r* &c) non reperitur selige depressissimum respectu dimen
sionum indefinitæ speciei (*x* vel *z* &c), dein alium terminum in quo sit ill
species radicalis selige, talem nempe ut progressio dimensionum utriusq
præfatæ speciei a termino priùs assumpto ad hunc terminum continuata, quàn
maximè potest descendat vel minimè ascendat.(27) Et siqui sint alij termin
quorum dimensiones cum hâc progressione ad arbitrium continuatâ con
veniant, eos etiam selige. Deniq; ex his selectis terminis tanquam nihilo æquali
bus quære valorem dictæ speciei radicalis,(28) et quotienti appone.

Cæterùm ut hæc regula magis elucescat, placuit insuper ope sequentis diagram
matis(29) exponere. Descripto angulo recto *BAC*, latera ejus *BA*, *AC* divido i
partes æquales, et inde normales erigo distribuentes angulare spatium in æquali
quadrata vel parallelogramma, quæ concipio denominata esse a dimensionibu
specierum *x* et *y*, prout vides in fig 1 inscriptas.(30) Deinde cùm æquatio aliqu

x^4	x^4y	x^4y^2	x^4y^3	x^4y^4
x^3	x^3y	x^3y^2	x^3y^3	x^3y^4
x^2	x^2y	x^2y^2	x^2y^3	x^2y^4
x	xy	xy^2	xy^3	xy^4
0	y	y^2	y^3	y^4

[fig 1]

[fig 2]

proponitur, parallelogramma singulis ejus terminis correspondentia insigni
notâ aliquâ, et Regulâ ad duo vel forte plura ex insignitis parallelogrammi

Fig. 1.2 Original Kruskal–Newton diagram, from *Methods of Series and Fluxions*, by Isaac Newton.

with the associated Kruskal–Newton diagram shown in Fig. 1.3. Terms in the polynomial are represented by dots. There are three possible placements of straight lines passing through two or more points with all remaining points above the line, as shown in the figure. Now for each line, treat the points above the line as a small perturbation, giving for line 1, with dominant terms $x, 1$,

$$x = 1 - [\epsilon x^2 + 2\epsilon^3 x^3], \tag{1.4}$$

with the terms in brackets associated with points above line 1, and thus small, since $x \simeq 1$ and $\epsilon \ll 1$. Now treat this equation iteratively.

In general an equation of the form

$$x = f(x), \tag{1.5}$$

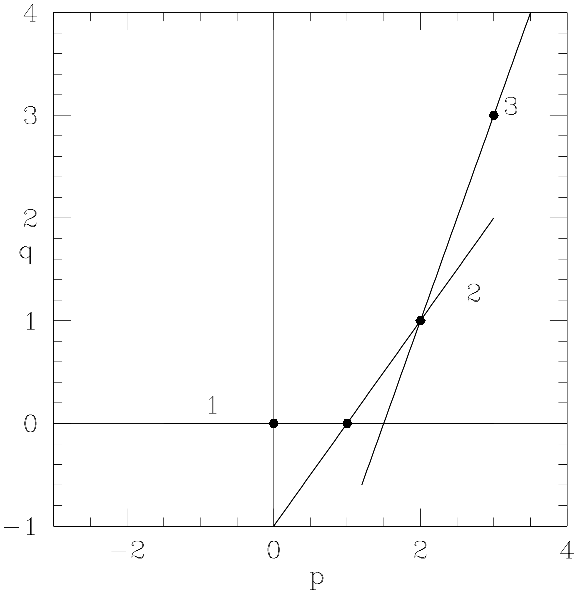


Fig. 1.3 Kruskal–Newton diagram showing lines of dominant balance.

is easily seen to converge to a fixed point x_f under the iteration scheme $x_{n+1} = f(x_n)$ provided the initial guess is sufficiently close to the solution and that $|f'(x_f)| < 1$. In this case $x_0 = 1$, and the fixed point cannot be far from this. We have $|f'| = |2\epsilon x + 6\epsilon^3 x^2| \simeq 2\epsilon$ for $x \simeq 1$, and thus the iteration converges. Convergence for $\epsilon = 0.3$ is shown in Fig. 1.4, with the function value shown as a downward sloping arc in red. Successive iterations are shown by the straight line segments starting at (1,0). After ten iterations the iterated value is correct to 15 places and converges much more rapidly if ϵ is smaller.

For line 2 the dominant terms are $x, \epsilon x^2$. By inspection $x \simeq 0$ is not a solution, so dividing by x we obtain

$$x = -\frac{1}{\epsilon} + \left[\frac{1 - 2\epsilon^3 x^3}{x\epsilon}\right], \tag{1.6}$$

giving the solution $x_0 \simeq -1/\epsilon$ and once again $|f'(x_0)| < 1$. For line 3 the dominant terms are $\epsilon x^2, 2\epsilon^3 x^3$ and we find

$$x = -\frac{1}{2\epsilon^2} + \left[\frac{1 - x}{2\epsilon^3 x^2}\right], \tag{1.7}$$

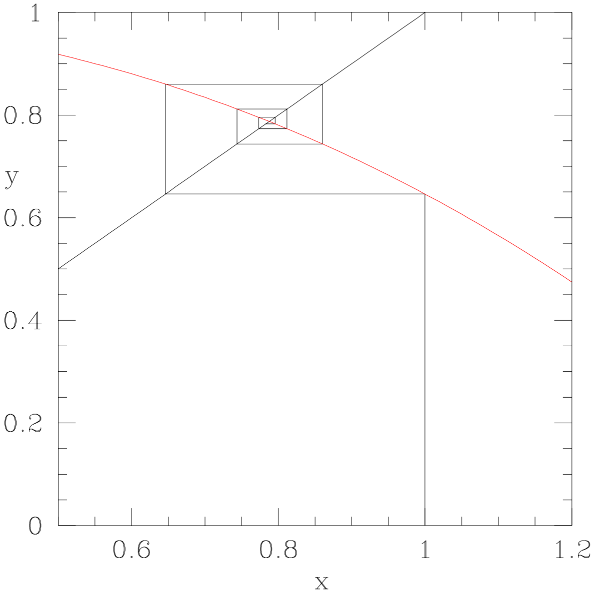


Fig. 1.4 Convergence of iteration of Eq. 1.4, $\epsilon = 0.3$.

giving a solution $x_0 \simeq -1/2\epsilon^2$ and once again $|f'(x_0)| < 1$. The original equation is a cubic polynomial and these three solutions give the complete set.

In general, to obtain a solution once a dominant balance is determined, simply treat the small terms as constant, and solve the resulting equation for x. The initial approximation x_0 is obtained by setting the small terms to zero, and evaluating them with x_n gives the iteration for x_{n+1}.

1.2.2 *Non polynomial form*

Consider the equation

$$2e^{-x} + \epsilon x^2 - 1 = 0. \tag{1.8}$$

The second derivative of this function is $2e^{-x} + 2\epsilon > 0$, and the function tends to $+\infty$ for $x \to \pm\infty$ and thus there are either two solutions or none. An associated Kruskal–Newton diagram is shown in Fig. 1.5. The points corresponding to the second and third terms are obvious. The placement of a point representing $2e^{-x}$ depends on the magnitude of x, since this term

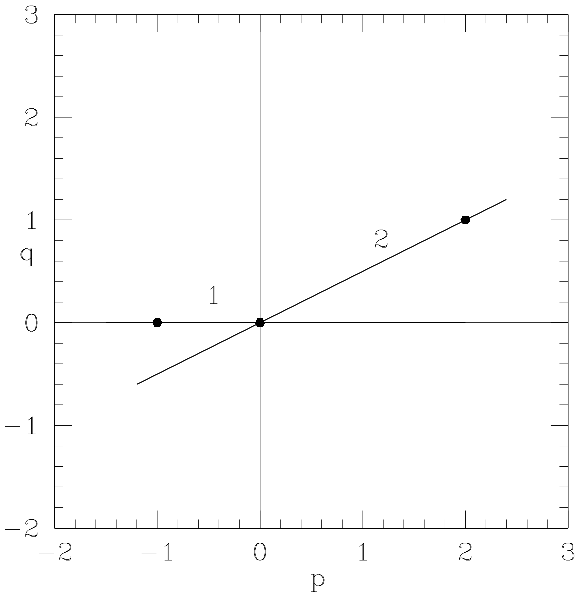


Fig. 1.5 Kruskal–Newton diagram showing lines of dominant balance for Eq. 1.8.

is not a simple power. In fact for various values of x it can assume the magnitude of x^p with p taking on any real value. The point must be along the line $q = 0$, and if it is placed to the right of the origin the resulting dominant balance would be $2e^{-x} + \epsilon x^2 \simeq 0$, which is impossible for x real. Thus place it to the left of the origin, giving the two dominant balances shown by the two lines in the figure. Line 1 gives the balance $2e^{-x} \simeq 1$ with $x_0 = ln(2)$. Treating the small term ϵx^2 perturbatively we find the iteration scheme

$$x_{n+1} = ln\left(\frac{2}{1 - \epsilon x_n^2}\right), \tag{1.9}$$

which can easily be seen to converge. For the second root line 2 gives the balance $\epsilon x^2 \simeq 1$ with $x_0 = 1/\sqrt{\epsilon}$. We then find the iteration scheme

$$x_{n+1} = \frac{1}{\sqrt{\epsilon}} - \frac{2e^{-x_n}}{\sqrt{\epsilon}(\sqrt{\epsilon}x_n + 1)}, \tag{1.10}$$

and these two iteration schemes give all possible roots.

Consider the equation

$$\frac{2cosx}{1+x} = 1 + 2\epsilon x^2 \tag{1.11}$$

where we are interested only in roots with $x > 0$. Multiplying by the denominator we have $1 + x - 2cosx + 2\epsilon x^2 + 2\epsilon x^3 = 0$. For $x > 0$ the only possible dominant balance is given by $2cosx = 1 + x$. It is important to realize that the initial guess x_0 need not be perfect, so write $cosx \simeq 1 - x^2/2$ giving $x_0 = (\sqrt{5} - 1)/2 \simeq 0.6$. The iteration $x_{n+1} = f(x_n)$ of the form

$$x_{n+1} = 2cosx_n - 1 - 2\epsilon x_n^2 - 2\epsilon x_n^3 \tag{1.12}$$

has $|f'(x_0)| \simeq 2sin(x_0) > 1$ so does not converge. If $f'(x_f)$ is not small, the convergence can often be improved by using the modified iteration

$$x_{n+1} = \frac{x_n}{2} + \frac{f(x_n)}{2} \tag{1.13}$$

and in this case produces a rapidly convergent sequence. Another way of finding the zeros of a function $F(x)$, used by Newton but discovered independently and first published by Joseph Raphson, known as the Newton–Raphson method, is given by using the first term of the Taylor series $F(x) = F(x_0) + F'(x_0)(x - x_0)$ and setting $F(x) = 0$, giving the iteration

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}. \tag{1.14}$$

Similarly, by choosing a second point near to x_0 and approximating the function as being linear, one finds the iteration

$$x_{n+1} = \frac{F(x_n)x_{n-1} - F(x_{n-1})x_n}{F(x_n) - F(x_{n-1})}. \tag{1.15}$$

Obviously, if root finding occupies a central part of a large numerical code it is important to find a reliable rapidly convergent scheme. Whatever iteration scheme is used, the Kruskal–Newton procedure gives a good initial guess x_0.

1.2.3 *Higher order*

Consider the equation

$$\epsilon x^5 + x^4 - 2\epsilon x^2 + \epsilon^2 = 0. \tag{1.16}$$

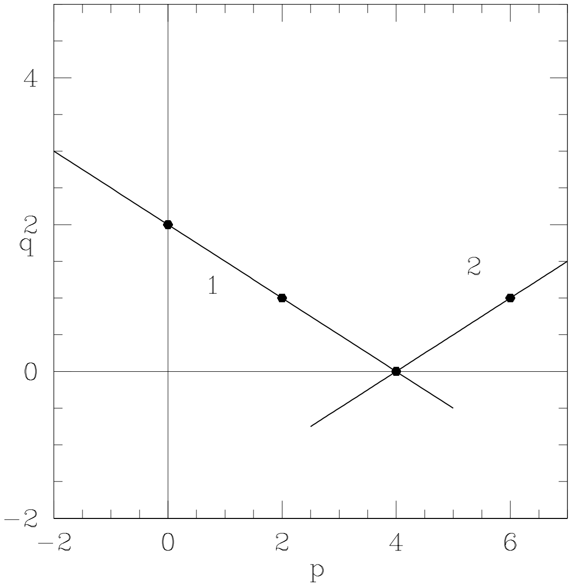


Fig. 1.6 Kruskal–Newton diagram showing lines of dominant balance for Eq. 1.16.

An associated Kruskal–Newton diagram is shown in Fig. 1.6, with the two dominant balances shown by the two lines. Line 1 gives the balance

$$x^4 - 2\epsilon x^2 + \epsilon^2 \simeq 0 \tag{1.17}$$

which is a quadratic equation in x^2, giving four iteration schemes with $x_0 = \pm\sqrt{\epsilon}$

$$x_{n+1} = \pm\sqrt{\epsilon \pm i\sqrt{\epsilon x_n^5}}. \tag{1.18}$$

Line 2 gives the dominant balance $\epsilon x^5 + x^4 \simeq 0$. Since $x \simeq 0$ is not a solution of the original equation, we find the iteration

$$x_{n+1} = -\frac{1}{\epsilon} + \frac{2\epsilon x_n^2 - \epsilon^2}{\epsilon x_n^4}, \tag{1.19}$$

with $x_0 = -1/\epsilon$, and these five iteration schemes give all possible roots.

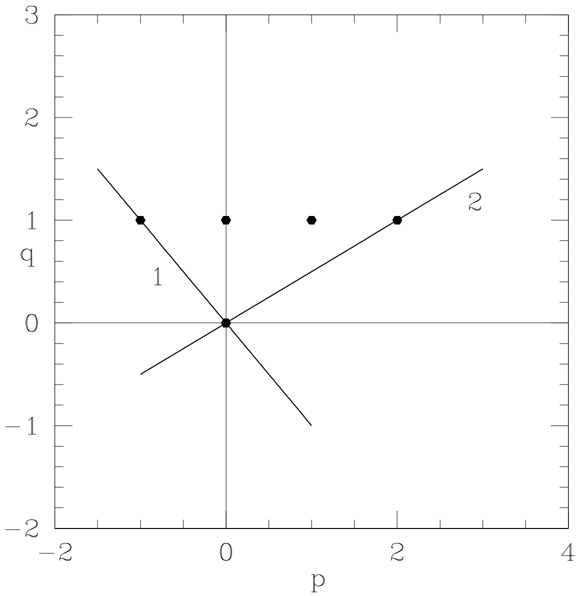


Fig. 1.7 Kruskal–Newton diagram with inaccessible points, Eq. 1.20.

1.2.4 *Hidden points*

It may often happen that some points in the Kruskal–Newton diagram are not accessible by any line giving dominant balance. An example is given in Fig. 1.7 by the equation

$$2\epsilon x^{-1} - 6\epsilon + \epsilon x + \epsilon x^2 - 1 = 0. \tag{1.20}$$

The only possible lines giving dominant balance are lines 1 and 2, and the points at $q = 1, p = 0, 1$ cannot be reached. Line 1 gives the iteration

$$x_{n+1} = 2\epsilon + \epsilon x_n^3 + \epsilon x_n^2 - 6\epsilon x_n, \tag{1.21}$$

with $x_0 = 2\epsilon$, and line 2 gives the iterations

$$x_{n+1} = \pm\sqrt{\left[\frac{1}{\epsilon} + 6 - x_n - \frac{2}{x_n}\right]} \tag{1.22}$$

with $x_0 = \pm 1/\sqrt{\epsilon}$, and these three iteration schemes give all possible roots.

As has been seen, these methods often reduce the order and complexity of a problem to a tractable level. It is easy to see that it is impossible to lose solutions, i.e. to come up with fewer iteration schemes than the order of the equation. Because it regularly finds all solutions this method is more powerful than perturbation theory, which loses some solutions if they happen not to be analytic functions of ϵ. The worst thing that can happen is that all points lie on a single line in the Kruskal–Newton diagram, in which case nothing at all is achieved by the method.

Note that in cases in which a solution behaves as $1/\epsilon$ or even some higher power, in the limit of $\epsilon \to 0$ the solution does not exist. Attempting a perturbation theory solution to the problem does not recover such a root, because it is not an analytic function of ϵ. The recovery of such non-analytic roots using perturbation methods is called singular perturbation theory. The Kruskal–Newton approach has no problem with such roots since it does not depend on analyticity of the root in ϵ in any way. Inverse and fractional powers regularly occur.

1.3 Problems

1. Use a Kruskal–Newton diagram to find dominant balances. Set up iteration schemes and show they converge.

$$\epsilon^2 x^4 - \epsilon x^2 - x - 1 = 0, \qquad \epsilon^2 x^3 + \epsilon x^2 - 2x + 4 = 0.$$

2. Use a Kruskal–Newton diagram to find dominant balances. Set up iteration schemes and show they converge.

$$\frac{\epsilon}{x^2} - \frac{2}{x} + x + 3\epsilon x^2 = 0, \qquad \epsilon^2 + 3\epsilon x - 4x^2 + \epsilon x^3 = 0.$$

3. Find the roots of the following equation to seven significant figures.

$$.01x^3 - x^2 + x + 6 = 0.$$

4. Find the real roots of $2/(1-\epsilon x^2) = e^x$ to order ϵ.

5. For $\epsilon << 1$ how many roots does the following equation have for $x > -1/\epsilon$, $k = O(1)$, $k > 1$?

$$(1+\epsilon x)^x = k.$$

Set up an iteration scheme, and show it converges.

6. Use a Kruskal–Newton diagram to find dominant balances. Set up an iteration scheme for each root and show it converges.

$$\epsilon x^3 + x^2 - 2x + 1 = 0.$$

7. Find the real roots of

$$\frac{sin(x)}{1-2x^2} = 1 + \epsilon x^2.$$

8. Find the real roots of

$$\frac{2cos(x)}{1+x^2} = 1 + \epsilon x^2.$$

9. Find the leading behavior and iteration schemes for all roots.

$$\epsilon x^8 - \epsilon^2 x^6 + x - 2 = 0, \qquad \epsilon^2 x^8 - \epsilon x^6 + x - 2 = 0.$$

10. Find the roots of the following equation to seven significant figures.

$$-\epsilon + x + x^2\epsilon^2 - \epsilon^2 x^3 = 0, \qquad \epsilon = .01.$$

11. Find the roots of the following equation to seven significant figures.

$$20 + x - \frac{1}{x} - x^2 = 0.$$

12. Zeros of the Wilkinson polynomial are given by

$$(x-1)(x-2)...(x-20) + \epsilon x^{19} = 0.$$

Is a Kruskal–Newton diagram of any help? Write a simple iteration scheme for the root approaching $x = n$ for $\epsilon \to 0$. To lowest order, show that the roots at 15, 16 collide for small ϵ. Estimate the value of ϵ and x for this collision. Do the iteration schemes converge at this x? Sometimes numerical evaluation of df/dx using $(f(x+d) - f(x))/d$ is easier than analytic differentiation!

13. Use a Kruskal–Newton diagram to find dominant balances. Set up iteration schemes and show they converge.

$$\epsilon^2 x^3 - x^2 + \epsilon x + 5 = 0$$
$$\epsilon + 3\epsilon x^2 - x - \epsilon^3 x^3 = 0$$

14. Find the real roots of the following equation to seven significant figures for $\epsilon = .01$.

$$\frac{1}{x} - \epsilon - 2x^2 + \epsilon x^3 = 0.$$

15. Find iteration schemes for solutions for the real roots of

$$2sin(x) = x + \epsilon x^3.$$

Find bounds on x_0 so that the iteration will converge, i.e. how accurate does x_0 have to be?

16. For $\epsilon << 1$ draw the Kruskal–Newton diagram for

$$-\epsilon^3 + 10\epsilon^2 x - 20\epsilon x^2 + x^4 = 0.$$

What are the orders of the solutions p, ie $x \sim \epsilon^p$? Can you find an iteration scheme?

How is the problem changed if $\epsilon = 0.1$, or $\epsilon = 10^{-2}$?