Section 1 Linear Algebra

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Let V be a real vector space of dimension at least 3 and let $T \in \mathrm{End}_{\mathbb{R}}(V)$. Prove that there is a non-zero subspace W of V, $W \neq V$, such that $T(W) \subseteq W$.

(*Indiana*)

Solution.

Make V into an $\mathbb{R}[\lambda]$-module by defining $\lambda \cdot v = T(v)$ for all $v \in V$. Thus for $\sum_{i=0}^{n} a_i \lambda^i \in \mathbb{R}[\lambda]$ and $v \in V$

$$\left(\sum_{i=0}^{n} a_i \lambda^i \right) \cdot v = \sum_{i=0}^{n} a_i T^i(v).$$

It is clear that a subspace W of V is an $\mathbb{R}[\lambda]$-submodule of V if and only if $T(W) \subseteq W$.

Now suppose V is a simple $\mathbb{R}[\lambda]$-module. Then $V \simeq \mathbb{R}[\lambda]/I$ for some maximal ideal of $\mathbb{R}[\lambda]$. Since $\mathbb{R}[\lambda]$ is a P.I.D., there exists an irreducible polynomial $f(\lambda)$ of $\mathbb{R}[\lambda]$ such that $I = (f(\lambda))$. So

$$3 \le \dim_{\mathbb{R}}(V) = \dim_{\mathbb{R}} \mathbb{R}[\lambda]/(f(\lambda)) = \deg f(\lambda).$$

This implies that we have an irreducible polynomial $f(\lambda)$ with degree ≥ 3 in $\mathbb{R}[\lambda]$. This is a contradiction. Hence V is not a simple $\mathbb{R}(\lambda)$-module, that is, there is a non-zero subspace W of V, $W \neq V$, such that $T(W) \subseteq W$.

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Let V be a finite dimensional vector space over a field K.

Let S be a linear transformation of V into itself. Let W be an invariant subspace of V (that is, $SW \subseteq W$). Let $m(t)$, $m_1(t)$, and $m_2(t)$ be the minimal polynomial of S as linear transformation of V, W and V/W respectively.

(a) Prove that $m(t)$ divides $m_1(t) \cdot m_2(t)$.

(b) Prove that if $m_1(t)$ and $m_2(t)$ are relatively prime, then

$$m(t) = m_1(t) \cdot m_2(t).$$

(c) Give an example of a case in which $m(t) \neq m_1(t) \cdot m_2(t)$.

(Indiana)

Solution.

As usual, V can be viewed as a $K[t]$-module via the linear transformation S. Since W is an S-invariant subspace of V, W is a $K[t]$-submodule of V. Then it is clear that $(m(t)) = \text{Ann}_{K[t]}V$, $(m_1(t)) = \text{Ann}_{K[t]}W$ and $(m_2(t)) = \text{Ann}_{K[t]}V/W$.

(a) Since

$$m_1(t) \cdot m_2(t) \cdot V \subseteq m_1(t) \cdot W = 0,$$
$$m_1(t) \cdot m_2(t) \in \text{Ann}_{K[t]}V = (m(t)).$$

Hence $m(t)$ divides $m_1(t) \cdot m_2(t)$.

(b) Since

$$m(t) \in \text{Ann}_{K[t]}V \subseteq \text{Ann}_{K[t]}W = (m_1(t)),$$

$m_1(t)$ divides $m(t)$. Similarly, $m_2(t)$ divides $m(t)$. Since $m_1(t)$ and $m_2(t)$ are relatively prime, $m_1(t) \cdot m_2(t)$ divides $m(t)$. Then we have $m(t) = m_1(t) \cdot m_2(t)$, since $m(t)$, $m_1(t)$ and $m_2(t)$ are all monic polynomials.

(c) Let W be a 2-dimensional vector space over the field Q of rational numbers and $S : W \to W$ be a linear transformation with minimal polynomial $t^2 + 1$. Let $V = W \oplus W$ and $S : V \to V$ be the natural extenaion of S to V. Then it is clear that $m(t) = m_1(t) = m_2(t) = t^2 + 1$. So $m(t) \neq m_1(t) \cdot m_2(t)$ in this example.

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Let V be a finite dimensional vector space over $\mathbb{R}$ and $T : V \to V$ be a linear transformation such that (a) the minimal polynomial of T is irreducible and (b) there exists a vector $v \in V$ such that $\{T^i v \mid i \geq 0\}$ spans V. Show that V doesn't have proper T-invariant subspace.

(Indiana)

Solution.

V can be viewed as a module over the polynomial ring $\mathbb{R}[\lambda]$ via $f(\lambda) \cdot x = f(T) \cdot (x)$, for any $f(\lambda) \in \mathbb{R}[\lambda]$ and $x \in V$. Then we have $V = \mathbb{R}[\lambda] \cdot v$, a cyclic module, since $\{T^i v \mid i \geq 0\}$ spans V by (b). Let $m(\lambda)$ be the minimal