Thus, when $p(x) = \frac{3}{5}x$,

$$\int_{-1}^{1}(p(x) - x^3)^2 dx$$

is minimal, and

$$\min_{p \in V} \int_{-1}^{1}(p(x) - x^3)^2 dx = \frac{8}{175}.$$

1111

Suppose that A is an $n \times n$ matrix and that

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \quad y^* = (y_1, \cdots, y_n).$$

Suppose that all the entries of A, x, and y are real.

(a) Show that there exist numbers a and b so that

$$\det(A + sxy^*) = a + bs.$$

(b) Show that if $\det(A) \neq 0$ then $a = \det(A)$ and $b = \det(A) \cdot y^* A^{-1} x$.

(c) Is it true that $a = 0$ if $\det(A) = 0$?

(*Courant Inst.*)

Solution.

(a) Directly,

$$\begin{aligned} & \det(A + sxy^*) \\ = \; & \det \begin{pmatrix} a_{11} + sx_1y_1 & \cdots & a_{1n} + sx_1y_n \\ \cdots & \cdots & \cdots \\ a_{n1} + sx_ny_1 & \cdots & a_{nn} + sx_ny_n \end{pmatrix} \\ = \; & \det \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \\ & + s \cdot \left(\det \begin{pmatrix} x_1y_1 & \cdots & x_1y_n \\ a_{21} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} + \det \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ x_2y_1 & \cdots & x_2y_n \\ a_{31} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \right. \end{aligned}$$

$$+\cdots+\det\left(\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n-1,1} & \cdots & a_{n-1,n} \\ a_n y_1 & \cdots & x_n y_n \end{pmatrix}\right).$$

Hence $\det(A + sxy^*) = a + bs$ for some a and b, for any s.

(b) Since for any s, $\det(A + sxy^*) = a + bs$ as in (a), we have $a = \det(A)$ and

$$\begin{aligned} b &= \det\begin{pmatrix} x_1y_1 & \cdots & x_1y_n \\ a_{21} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} + \det\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ x_2y_1 & \cdots & x_2y_n \\ a_{31} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \\ &\quad + \cdots + \det\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n-11} & \cdots & a_{n-1n} \\ x_ny_1 & \cdots & x_ny_n \end{pmatrix} \\ &= \sum_{j=1}^{n} x_1y_jA_{1j} + \sum_{j=1}^{n} x_2y_jA_{2j} + \cdots + \sum_{j=1}^{n} x_ny_jA_{nj} \\ &= \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{n} y_jA_{ij}\right), \end{aligned}$$

where A_{ij} is the algebraic cofactor of a_{ij}. If $\det A \neq 0$,

$$A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} A_{11} & \cdots & A_{n1} \\ \cdots & \cdots & \cdots \\ A_{1n} & \cdots & A_{nn} \end{pmatrix}.$$

Thus

$$\begin{aligned} b &= \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{n} y_jA_{ij}\right) \\ &= y^* \cdot \begin{pmatrix} A_{11} & \cdots & A_{n1} \\ \cdots & \cdots & \cdots \\ A_{1n} & \cdots & A_{nn} \end{pmatrix} \cdot x \\ &= y^*(\det(A) \cdot A^{-1})x \\ &= \det(A) \cdot y^*A^{-1}x. \end{aligned}$$

(c) If $\det(A) = 0$, $a = \det(A) = 0$.