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Let A be an $n \times n$ real matrix with distinct (possibly complex) eigenvalues, $\lambda_1, \lambda_2, \cdots, \lambda_n$, and corresponding eigenvector $v_1, v_2, \cdots, v_n$. Assume that $\lambda_1 = 1$ and that $|\lambda_j| < 1$ for $2 \leq j \leq n$. Prove that $\lim\limits_{n\to\infty} A^n v$ exists. Define $T : \mathbb{C}^n \to \mathbb{C}^n$ by $T(v) = \lim\limits_{n\to\infty} A^n v$. Find the dimensions of the kernel and image of T and give basis for both.

(*Courant Inst.*)

Solution.

Let $P = (v_1, v_2, \cdots, v_n)$. Since $\lambda_1, \lambda_2, \cdots, \lambda_n$ are distinct, P is an invertible matrix in $M_n(\mathbb{C})$ and $P^{-1}AP = \mathrm{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$.

For any $v \in \mathbb{C}^n$,

$$\begin{aligned} \lim_{n\to\infty} A^n v &= \lim_{n\to\infty} P\mathrm{diag}\{\lambda_1^n, \lambda_2^n, \cdots, \lambda_n^n\} \cdot P^{-1} \cdot v \\ &= P \cdot \mathrm{diag}\{1, 0, \cdots, 0\} \cdot P^{-1} \cdot v \end{aligned}$$

(Since $\lim\limits_{n\to\infty} (X_n Y_n) = \lim\limits_{n\to\infty} X_n \cdot \lim\limits_{n\to\infty} Y_n$). Let $(e_1, e_2, \cdots, e_n)$ be the standard orthonormal basis of $\mathbb{C}^n$. Then the matrix of T with respect to this basis is $P'^{-1}\mathrm{diag}(1, 0, \cdots, 0) \cdot P'$. Let

$$\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = P' \cdot \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}.$$

Then $(f_1, f_2, \cdots, f_n)$ is a basis of $\mathbb{C}^n$ and

$$\mathrm{diag}(1, 0, \cdots, 0) = P' \cdot P'^{-1} \cdot \mathrm{diag}(1, 0, \cdots, 0) P' P'^{-1}$$

is the matrix of T with respect to the basis $(f_1, f_2, \cdots, f_n)$. Hence $\{f_1\}$ is a basis of $\mathrm{Im}(T)$, $\{f_2, \cdots, f_n\}$ is a basis of $\ker(T)$, and $\dim(\mathrm{Im}(T)) = 1$, $\dim(\ker(T)) = n - 1$.

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Let A be a matrix. Define

$$\sin(A) = A - \frac{1}{3!}A^3 + \frac{1}{5!}A^5 - \cdots.$$

For

$$A = \frac{\pi}{4}\begin{pmatrix} 7 & -3 \\ -3 & 7 \end{pmatrix},$$