

## Chapter 1

# The Basics of $C^*$ -algebras

### 1.1 Banach algebras

**Definition 1.1.1** A *normed algebra* is a complex algebra  $A$  which is a normed space, and the norm satisfies

$$\|ab\| \leq \|a\|\|b\| \text{ for all } a, b \in A.$$

If  $A$  (with this norm) is complete, then  $A$  is called a *Banach algebra*.

Every closed subalgebra of a Banach algebra is itself a Banach algebra.

**Example 1.1.2** Let  $\mathbb{C}$  be the complex field. Then  $\mathbb{C}$  is a Banach algebra. Let  $X$  be a compact Hausdorff space and  $C(X)$  the set of continuous functions on  $X$ .  $C(X)$  is a complex algebra with pointwise operations. With  $\|f\| = \sup_{x \in X} |f(x)|$ ,  $C(X)$  is a Banach algebra.

**Example 1.1.3** Let  $M_n$  be the algebra of  $n \times n$  complex matrices. By identifying  $M_n$  with  $B(\mathbb{C}^n)$ , the set of all (bounded) linear maps from the  $n$ -dimensional Hilbert space  $\mathbb{C}^n$  to  $\mathbb{C}^n$ , with operator norm, i.e.,  $\|x\| = \sup_{\xi \in \mathbb{C}^n, \|\xi\| \leq 1} \|x(\xi)\|$ , we see that  $M_n$  is a Banach algebra.

**Example 1.1.4** The set of continuous functions  $A(\mathbf{D})$  on the closed unit disk  $\mathbf{D}$  in the plane which are analytic on the interior is a closed subalgebra of  $C(\mathbf{D})$ . Therefore,  $A(\mathbf{D})$  is a Banach algebra.

**Example 1.1.5** Let  $X$  be a Banach space and  $B(X)$  be the set of all bounded linear operators on  $X$ . If  $T, L \in B(X)$ , define  $TL = T \circ L$ . Then  $B(X)$  is a complex algebra. With operator norm,  $B(X)$  is a Banach algebra.

A *commutative* Banach algebra is a Banach algebra  $A$  with the property that  $ab = ba$  for all  $a, b \in A$ . Examples 1.1.2 and 1.1.4 are of commutative Banach algebras while Example 1.1.3 1.1.5 are not commutative.

**Definition 1.1.6** In a unital algebra, an element  $a \in A$  is called *invertible* if there is an element  $b \in A$  such that  $ab = ba = 1$ . In this case  $b$  is unique and written  $a^{-1}$ . The set

$$GL(A) = \{a \in A : a \text{ is invertible}\}$$

is a group under multiplication.

We define the *spectrum* of an element  $a$  to be the set

$$\text{sp}(a) = \text{sp}_A(a) = \{\lambda \in \mathbb{C} : \lambda 1 - a \notin GL(A)\}.$$

Whenever there is no confusion, we will write  $\lambda 1$  simply as  $\lambda$ .

The complement of the spectrum is called the *resolvent* and  $R(\lambda) = (\lambda - a)^{-1}$  is the *resolvent function*.

**Example 1.1.7** Let  $A = C(X)$  be as in 1.1.2. Then  $\text{sp}(f) = f(X)$  for all  $f \in A$ . In other words, the spectrum of  $f$  is the range of  $f$ .

Let  $A = M_n$ . If  $a = (a_{ij}) \in A$ , then the reader can check that  $\text{sp}(a)$  is the set of eigenvalues of the matrix  $a$ .

**Proposition 1.1.8** For any  $a$  and  $b$  in  $A$ ,

$$\text{sp}(ab) \setminus \{0\} = \text{sp}(ba) \setminus \{0\}.$$

**Proof.** If  $\lambda \notin \text{sp}(ab)$  and  $\lambda \neq 0$ , then there is  $c \in A$  such that

$$c(\lambda - ab) = (\lambda - ab)c = 1.$$

Thus  $c(ab) = \lambda c - 1 = (ab)c$ . So we compute that

$$\begin{aligned} (1 + bca)(\lambda - ba) &= \lambda - ba + \lambda bca - bcaba \\ &= \lambda - ba + b(\lambda - ab)ca = \lambda - ba + ba = \lambda, \end{aligned}$$

which shows that  $\lambda^{-1}(1 + bca)$  is the inverse of  $\lambda - ba$ . Hence  $\lambda \notin \text{sp}(ba)$  and  $\text{sp}(ba) \setminus \{0\} \subset \text{sp}(ab) \setminus \{0\}$ .  $\square$

**Definition 1.1.9** A Banach algebra  $A$  is said to be *unital* if it admits a unit  $1$  and  $\|1\| = 1$ . Banach algebras in 1.1.2, 1.1.3 and 1.1.4 are unital.

**Lemma 1.1.10** *Let  $A$  be a unital Banach algebra and  $a$  be an element of  $A$  such that  $\|1 - a\| < 1$ . Then  $a \in GL(A)$  and*

$$a^{-1} = \sum_{n=0}^{\infty} (1 - a)^n.$$

Moreover,  $\|a^{-1}\| \leq \frac{1}{1 - \|1 - a\|}$  and  $\|1 - a^{-1}\| \leq \frac{\|1 - a\|}{1 - \|1 - a\|}$ .

**Proof.** Since

$$\sum_{n=0}^{\infty} \|(1 - a)^n\| \leq \sum_{n=0}^{\infty} \|1 - a\|^n = \frac{1}{(1 - \|1 - a\|)} < \infty,$$

the series  $\sum_{n=0}^{\infty} (1 - a)^n$  is convergent. Let  $b$  be its limit in  $A$ . Then  $\|b\| \leq \frac{1}{(1 - \|1 - a\|)}$  and

$$\|1 - b\| \leq \sum_{n=1}^{\infty} \|1 - a\|^n = \frac{\|1 - a\|}{1 - \|1 - a\|}.$$

One verifies

$$a \left( \sum_{n=0}^k (1 - a)^n \right) = (1 - (1 - a)) \left( \sum_{n=0}^k (1 - a)^n \right) = 1 - (1 - a)^{k+1}$$

and that it converges to  $ab = ba = 1$  as  $k \rightarrow \infty$ . Hence  $b$  is the inverse of  $a$ .  $\square$

**Definition 1.1.11** A function  $f$  from an open subset  $\Omega \subset \mathbb{C}$  to a Banach algebra is said to be *analytic*, if for any  $\lambda_0 \in \Omega$  there is an open neighborhood  $O(\lambda_0)$  such that  $f(\lambda) = \sum_{n=0}^{\infty} a_n(\lambda - \lambda_0)^n$  converges for every  $\lambda \in O(\lambda_0)$ . To include the case that  $\lambda_0 = \infty$ , we say  $f$  is analytic at infinity if  $f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^{-n}$  for all  $\lambda$  in a neighborhood of infinity.

**Theorem 1.1.12** *In any unital Banach algebra  $A$ , the spectrum of each  $a \in A$  is a non-empty compact subset, and the resolvent function is analytic on  $\mathbb{C} \setminus \text{sp}(a)$ .*

**Proof.** If  $|\lambda| > \|a\|$ , then  $\|\lambda^{-n} a^n\| \leq \left(\frac{\|a\|}{\lambda}\right)^n$ . So the series

$$\sum_{n=0}^{\infty} \frac{a^n}{\lambda^{n+1}}$$

converges (in norm). Similarly to 1.1.10,

$$(\lambda - a) \sum_{n=0}^k \frac{a^n}{\lambda^{n+1}} = 1 - \left( \frac{a^{k+1}}{\lambda^{k+1}} \right)$$

which converges to 1. This shows that  $\sup_{\lambda \in \text{sp}(a)} |\lambda| \leq \|a\|$  and  $R(\lambda)$  is analytic in  $\{\lambda : |\lambda| > \|a\|\}$ . Moreover,

$$\lim_{|\lambda| \rightarrow \infty} \|R(\lambda)\| \leq \lim_{|\lambda| \rightarrow \infty} \frac{|\lambda|^{-1}}{1 - \left( \frac{\|a\|}{|\lambda|} \right)} = 0. \quad (\text{e 1.1})$$

Similarly, if  $\lambda_0 - a$  is invertible and  $|\lambda - \lambda_0| < \frac{1}{\|(\lambda_0 - a)^{-1}\|}$ , then

$$(\lambda - a)^{-1} = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n ((\lambda_0 - a)^{-1})^{n+1}.$$

This also shows that the resolvent is open. Since  $\text{sp}(a)$  has been shown to be bounded,  $\text{sp}(a)$  is compact. We have also shown that the resolvent function is analytic on the complement of the spectrum.

In particular,  $f(R(\lambda))$  is a (scalar) analytic function for every bounded linear functional  $f \in A^*$ . If  $\text{sp}(a)$  were empty, then  $f(R(\lambda))$  would be an entire function for every  $f \in A^*$ . However, (e 1.1) shows that  $f(R(\lambda))$  is bounded on the plane. Thus Liouville's theorem implies that  $f(R(\lambda))$  is a constant. But (e 1.1) also implies that  $f(R(\lambda)) = 0$ . So, by the Hahn-Banach theorem,  $R(\lambda) = 0$ . This is a contradiction. Hence  $\text{sp}(a)$  is not empty.  $\square$

**Corollary 1.1.13** *The only simple commutative unital Banach algebra is  $\mathbb{C}$ .*

**Proof.** Suppose that  $A$  is a unital commutative Banach algebra and  $a \in A$  is not a scalar. Let  $\lambda \in \text{sp}(a)$ . Set  $I = \overline{(a - \lambda)A}$ . Then  $I$  is clearly a closed ideal of  $A$ . No element of the form  $(a - \lambda)b$  is invertible in the commutative Banach algebra  $A$ . By 1.1.10,

$$\|(a - \lambda)b - 1\| \geq 1.$$

So  $1 \notin I$  and  $I$  is proper. Therefore, if  $A$  is simple,  $a$  must be a scalar, whence  $A = \mathbb{C}$ .  $\square$

**Lemma 1.1.14** *If  $p$  is a polynomial and  $a$  is an element of a unital Banach algebra  $A$ , then*

$$\text{sp}(p(a)) = p(\text{sp}(a)).$$

**Proof.** We may assume that  $p$  is not constant. If  $\lambda \in \mathbb{C}$ , there are  $c, \beta_1, \dots, \beta_n \in \mathbb{C}$  such that

$$p(z) - \lambda = c \prod_{i=1}^n (z - \beta_i),$$

and therefore

$$p(a) - \lambda = c \prod_{i=1}^n (a - \beta_i).$$

It is clear that  $p(a) - \lambda$  is invertible if and only if each  $a - \beta_i$  is. It follows that  $\lambda \in \text{sp}(p(a))$  if and only if  $\lambda = p(\alpha)$  for some  $\alpha \in \text{sp}(a)$ . Thus  $\text{sp}(p(a)) = p(\text{sp}(a))$ .  $\square$

**Definition 1.1.15** Let  $A$  be a unital Banach algebra. If  $a \in A$ , its *spectral radius* is defined to be

$$r(a) = \sup_{\lambda \in \text{sp}(a)} |\lambda|.$$

**Theorem 1.1.16** If  $a$  is an element in a unital Banach algebra  $A$ , then

$$r(a) = \lim_{n \rightarrow \infty} \|a^n\|^{1/n}.$$

**Proof.** If  $\lambda \in \text{sp}(a)$ , then  $\lambda^n \in \text{sp}(a^n)$  by 1.1.14, so  $|\lambda^n| \leq \|a^n\|$ . Therefore,  $r(a) \leq \liminf_{n \rightarrow \infty} \|a^n\|^{1/n}$ . Let  $\Omega$  be the open disk in  $\mathbb{C}$  with center 0 and radius  $1/r(a)$  (or  $\infty$  if  $r(a) = 0$ ). If  $\lambda \in \Omega$ , then  $1 - \lambda a \in GL(A)$ . If  $f \in A^*$ , then  $f((1 - \lambda a)^{-1})$  is analytic. There are unique complex numbers  $z_n$  such that

$$f((1 - \lambda a)^{-1}) = \sum_{n=0}^{\infty} z_n \lambda^n \quad (\lambda \in \Omega).$$

However, if  $|\lambda| < 1/\|a\| \leq 1/r(a)$ , then  $\|\lambda a\| < 1$ , so  $(1 - \lambda a)^{-1} = \sum_{n=0}^{\infty} \lambda^n a^n$ , and therefore,  $f((1 - \lambda a)^{-1}) = \sum_{n=0}^{\infty} \lambda^n f(a^n)$ . It follows that  $z_n = f(a^n)$  for all  $n \geq 0$ . Hence the sequence  $\{\lambda^n f(a^n)\}$  converges to zero for each  $\lambda \in \Omega$ , and therefore is bounded. Since this is true for every  $f \in A^*$ , by the principle of uniform boundedness,  $\{\lambda^n a^n\}$  is a bounded sequence. So we may assume that  $|\lambda^n| \|a^n\| \leq M$  for all  $n \geq 0$  and for some positive number  $M$ . Hence

$$\|a^n\|^{1/n} \leq \frac{M^{1/n}}{|\lambda|}, \quad n = 0, 1, \dots$$

Consequently,

$$\limsup_{n \rightarrow \infty} \|a^n\|^{1/n} \leq 1/|\lambda|.$$

This implies that if  $r(a) < \frac{1}{|\lambda|}$ , then  $\limsup_{n \rightarrow \infty} \|a^n\|^{1/n} \leq 1/|\lambda|$ . It follows that  $\limsup_{n \rightarrow \infty} \|a^n\|^{1/n} \leq r(a)$ . From what we have shown at the beginning of this proof, we obtain

$$\limsup_{n \rightarrow \infty} \|a^n\|^{1/n} \leq r(a) \leq \liminf_{n \rightarrow \infty} \|a^n\|^{1/n}.$$

This implies that  $r(a) = \lim_{n \rightarrow \infty} \|a^n\|^{1/n}$ . □

**Example 1.1.17** Let  $A = M_3$  and  $a = \begin{pmatrix} 1/2 & 1 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}$ . Then  $\|a\| \geq 1$  and  $\text{sp}(a) = \{1/2, 1/3\}$ . So  $r(a) = 1/2$ . It follows that  $\|a^n\|^{1/n} \rightarrow 1/2$ .

Let  $T \in B(L^2([0, 1]))$  be a bounded linear operator defined by

$$T(f) = \int_0^t f(x) dx.$$

The reader can compute that  $\|T^n\| \leq \frac{1}{n!}$ . Hence  $r(T) = 0$ . Note that  $T \neq 0$  (see Exercise 1.11.3).

We now establish the holomorphic functional calculus for elements in Banach algebras (1.1.19). The first application appears in 1.2.9. Later, we will establish continuous functional calculus for commutative  $C^*$ -algebras (1.3.5) and Borel functional calculus for normal elements in von Neumann algebras (1.8.5).

**Definition 1.1.18** Let  $x$  be a fixed element in a unital Banach algebra  $A$ . Let  $f$  be a holomorphic function in an open neighborhood  $O_f$  of  $\text{sp}(x)$ , and  $C$  be a smooth simple closed curve in  $O_f$  enclosing  $\text{sp}(x)$ . We assign the positive orientation to  $C$  as in complex analysis. For each  $\phi \in A^*$ , we consider a continuous function which maps  $\lambda$  to  $f(\lambda)\phi((\lambda - x)^{-1})$  on the curve  $C$ . Set

$$L(\phi) = \frac{1}{2\pi i} \int_C f(\lambda)\phi((\lambda - x)^{-1}) d\lambda.$$

The map  $\phi \mapsto L(\phi)$  is a linear functional on  $A^*$  and

$$|L(\phi)| \leq \frac{l}{2\pi} \|\phi\| \sup\{|f(\lambda)| \|(\lambda - x)^{-1}\| : \lambda \in C\},$$

where  $l$  is the length of the curve  $C$ . Hence there exists an  $F \in A^{**}$  such that  $F(\phi) = L(\phi)$ .

On the other hand, the function  $\lambda \mapsto f(\lambda)(\lambda - x)^{-1}$  is a continuous function from  $C$  into  $A$ . So the limit of

$$\sum_{i=0}^n f(\lambda_i)(\lambda_i - x)^{-1}(\lambda_i - \lambda_{i+1}) \quad (\text{as } \max_i |\lambda_i - \lambda_{i+1}| \rightarrow 0),$$

where  $\{\lambda_0, \dots, \lambda_n, \lambda_{n+1} = \lambda_0\}$  is a partition of the curve  $C$ , converges in norm in  $A$  to  $y$ . By the continuity of  $\phi$ , we know that  $\phi(y) = L(\phi) = F(\phi)$  for all  $\phi \in A^*$ . Hence  $F \in A$ . By Cauchy's theorem,  $F$  does not depend on the choice of the curve  $C$ , but only on the function  $f$ . Therefore we may denote  $F$  by  $f(x)$  and write

$$f(x) = \frac{1}{2\pi i} \int_C f(\lambda)(\lambda - x)^{-1} d\lambda.$$

We denote by  $\text{Hol}(\text{sp}(x))$  the algebra of all functions which are holomorphic in a neighborhood of  $\text{sp}(x)$ .

**Theorem 1.1.19** *Fix an element  $x$  in a unital Banach algebra  $A$ . The map  $f \mapsto f(x)$  from  $\text{Hol}(\text{sp}(x))$  into  $A$  is a homomorphism which sends the constant function 1 to the identity of  $A$  and the identity function on  $\mathbb{C}$  to the element  $x$ . If  $f(z) = \sum_{n=0}^{\infty} c_n z^n$  in a neighborhood of  $\text{sp}(x)$ , then  $f(x) = \sum_{n=0}^{\infty} c_n x^n$ .*

**Proof.** Linearity is clear. Let  $f$  and  $g$  be functions holomorphic in neighborhoods  $O_f$  and  $O_g$  of  $\text{sp}(x)$ , respectively. Set  $O = O_f \cap O_g$  and let  $C_i, i = 1, 2$  be smooth simple closed curves in  $O$  enclosing  $\text{sp}(x)$  such that  $C_1$  lies completely inside the curve  $C_2$ . Then

$$\begin{aligned} f(x)g(x) &= \left(\frac{1}{2\pi i} \int_{C_1} f(\lambda)(\lambda - x)^{-1} d\lambda\right) \left(\frac{1}{2\pi i} \int_{C_2} g(z)(z - x)^{-1} dz\right) \\ &= -\frac{1}{4\pi^2} \int_{C_1} \left[ \int_{C_2} f(\lambda)g(z)(\lambda - x)^{-1}(z - x)^{-1} dz \right] d\lambda \\ &= -\frac{1}{4\pi^2} \int_{C_1} \int_{C_2} f(\lambda)g(z) \frac{[(z - x)^{-1} - (\lambda - x)^{-1}]}{\lambda - z} dz d\lambda \\ &= \frac{1}{2\pi i} \int_{C_1} f(\lambda)(\lambda - x)^{-1} \left(\frac{1}{2\pi i} \int_{C_2} \frac{g(z)}{z - \lambda} dz\right) d\lambda \\ &\quad + \frac{1}{2\pi i} \int_{C_2} g(z)(z - x)^{-1} \left(\frac{1}{2\pi i} \int_{C_1} \frac{f(\lambda)}{\lambda - z} d\lambda\right) dz \end{aligned}$$

$$= \frac{1}{2\pi i} \int_{C_1} f(\lambda)g(\lambda)(\lambda - x)^{-1}d\lambda = (f \cdot g)(x).$$

The second to last equality holds because  $\frac{1}{2\pi i} \int_{C_2} \frac{g(z)}{z-\lambda} dz = g(\lambda)$  (by the Cauchy formula) and because  $\frac{f(\lambda)}{\lambda-z}$  is holomorphic inside the curve  $C_2$  if  $\lambda \in C_1$  (so that  $\int_{C_1} \frac{f(\lambda)}{\lambda-z} d\lambda = 0$ ).

To complete the proof, pick a circle  $C$  with center at 0 and large radius. We note that

$$\lambda^{n-1}(1 - \lambda^{-1}x)^{-1} = \lambda^{n-1} \sum_{k=0}^{\infty} x^k \lambda^{-k} = \sum_{k=0}^{\infty} x^k \lambda^{n-k-1},$$

where the convergence is in norm and is uniform on  $C$ . Therefore

$$\int_C x^k \lambda^{n-k-1} d\lambda = \left( \int_C \lambda^{n-k-1} d\lambda \right) x^k = 0 \cdot x^k = 0 \quad (\text{e 1.2})$$

unless  $k = n$ , in which case the integral is  $2\pi i x^n$ . This implies that

$$x^n = \frac{1}{2\pi i} \int_C \lambda^n (\lambda - x)^{-1} d\lambda. \quad (\text{e 1.3})$$

Now suppose that  $f(z) = \sum_{n=0}^{\infty} c_n z^n$  in a neighborhood of  $\text{sp}(x)$ . Then it converges in an open disk with center 0. Let  $C$  be a circle with center at 0 contained in the open disk. Then the series converges uniformly on  $C$ , so from (e 1.3),

$$f(x) = \frac{1}{2\pi i} \int_C f(z)(z-x)^{-1} dz = \sum_{n=0}^{\infty} c_n \left( \frac{1}{2\pi i} \int_C z^n (z-x)^{-1} dz \right) = \sum_{n=0}^{\infty} c_n x^n. \quad \square$$

**Proposition 1.1.20** *If  $I$  is a closed ideal in a Banach algebra, then  $A/I$  is a Banach algebra with the quotient norm*

$$\|\bar{a}\| = \|a + I\| = \inf_{b \in I} \|a + b\|.$$

**Proof.** It is well known that  $A/I$ , as a normed space is complete. It remains to show that  $A/I$  is a normed algebra. Let  $\varepsilon > 0$  and  $a, b \in A$ . Then, there are  $i, j \in I$  such that

$$\|a + i\| < \|a + I\| + \varepsilon \quad \text{and} \quad \|b + j\| < \|b + I\| + \varepsilon.$$

Hence, for  $c = ib + aj + ij \in I$ ,

$$\|ab + c\| \leq \|a + i\| \|b + j\| < (\|a + I\| + \varepsilon)(\|b + I\| + \varepsilon).$$

Thus,

$$\|ab + I\| \leq (\|a + I\| + \varepsilon)(\|b + I\| + \varepsilon).$$

Letting  $\varepsilon \rightarrow 0$ , we obtain

$$\|ab + I\| \leq \|a + I\| \|b + I\|.$$

In other words,  $A/I$  is a normed algebra. □

## 1.2 $C^*$ -algebras

**Definition 1.2.1** An algebra  $A$  is called a  $*$ -algebra if it is a complex algebra with a conjugate linear involution  $*$  which is an anti-isomorphism, i.e., for any  $a, b \in A$  and  $\alpha \in \mathbb{C}$ ,

$$(a + b)^* = a^* + b^*, \quad (\alpha a)^* = \bar{\alpha} a^*, \quad a^{**} = a \quad \text{and} \quad (ab)^* = b^* a^*.$$

If  $a \in A$ , then  $a^*$  is called the *adjoint* of  $a$ . Let  $A$  be a  $*$ -algebra which is also a normed algebra. A norm on  $A$  that satisfies

$$\|a^* a\| = \|a\|^2$$

for all  $a \in A$  is called a  $C^*$ -norm. If, with this norm,  $A$  is complete, then  $A$  is called a  $C^*$ -algebra.

Since  $\|x^* x\| \leq \|x^*\| \|x\|$  we have  $\|x\| \leq \|x^*\|$  for all  $x \in A$ . Thus  $\|x^*\| = \|x\|$ .

A closed  $*$ -subalgebra of a  $C^*$ -algebra is also a  $C^*$ -algebra. Such a  $*$ -subalgebra will be called a  $C^*$ -subalgebra.

**Example 1.2.2** (a) Let  $A$  be as in 1.1.3 and let  $a = (a_{ij}) \in A$ . Define  $a^* = (\bar{a}_{ji})$ . Then  $A$  is a  $C^*$ -algebra.

(b) Let  $X$  be a locally compact Hausdorff space and  $C_0(X)$  be the set of all continuous functions vanishing at infinity. Define  $f^*(t) = \overline{f(t)}$  (for  $t \in X$ ). Then  $C_0(X)$  becomes a  $*$ -algebra. With  $\|f\| = \sup_{t \in X} |f(t)|$ ,  $C_0(X)$  is a  $C^*$ -algebra.  $C_0(X)$  is unital if and only if  $X$  is compact.

**Example 1.2.3** Let  $X$  be a compact Hausdorff space and  $\mathcal{B}(X)$  be the set of all bounded Borel functions on  $X$ . With  $\|f\| = \sup_{x \in X} |f(x)|$  and  $f^*(x) = \overline{f(x)}$ ,  $\mathcal{B}(X)$  becomes a  $C^*$ -algebra. For any complex Borel measure