

Contents

(Omitted sections consist of questions. A few questions of special interest are listed.)

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