# PREFACE

The notion of minimal surfaces was initiated by J. L. Lagrange in 1762. Lagrange studied a graph $M$ defined by a smooth function $z = f(x, y)$ in 3-dimensional Euclidean space $\mathbb{R}^3$ endowed with the standard orthogonal coordinates $(x, y, z)$. Let $D \subset M$ be any domain with the boundary $\partial D$. If every such $D$ has least area among all the comparable surfaces with the same boundary $\partial D$, the function $f$ satisfies the partial differential equation:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial}{\partial f}\partial y\right)^2}\right) + \frac{\partial}{\partial y}\left(\frac{\frac{\partial f}{\partial y}}{\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}}\right) = 0, \tag{0.1}$$

or equivalently

$$(1+f_y^2)\,f_{xx} - 2\,f_x\,f_y\,f_{xy} + (1+f_x^2)\,f_{yy} = 0, \tag{0.2}$$

here subscripts denote partial derivatives with respect to indicated variables. We thus obtain a quasilinear PDE, elliptic for all values of the arguments appearing in the coefficients. Lagrange noted that linear functions are trivial solutions to (0.2) and their graphs are planes..

In 1776 J. B. Meusnier pointed out the geometric interpretation of the equation (0.2), that the mean curvature $H$ of $M$ in $\mathbb{R}^3$ vanishes. He also discovered two nonlinear solutions to the equation, whose graphs are catenoid and helicoid. Further particular solutions were later discovered, and in 1866 K. Weierstrass obtained his representation formula for minimal surfaces in terms of holomorphic data. From that formula, one obtains general families of solutions to the equation.

In 1915 S. Bernstein proved his celebrated theorem that the only entire minimal graphs in Euclidean 3-space are planes. More precisely, any solution to (0.2) over the entire plane is a linear function. The ensuing efforts to generalize Bernstein's theorem to higher dimension led to profound developments in analysis and in geometric measure theory.

From another point of view, the Belgian physicist J. Plateau made many remarkable experiments. Dipping a frame of thin wire into a soap solution and then skillfully removing it out, one obtained one or more soap films bounded by the wire . He showed in 1847 using laws of surface tension that every soap film represents a surface which has least area under small deformations. Thus such films are determined as solutions of minimal surface equation. The celebrated "Problem of Plateau" is to determine whether every simple closed "wire frame" admits a "soap film". Satisfactory solutions were given by J. Douglas and T. Rado in 1930. For this achievement, Douglas was awarded the "Fields Medal".

The Bernstein problem and Plateau problem are central topics in the theory of minimal surfaces. In the present book we present the Douglas - Rado solution to the Plateau problem, but our main emphasis will be directed to the Bernstein problem and its new developments in various directions.

We also introduce some related topics: among them submanifolds with parallel mean curvature, Weierstrass type representation for surfaces of mean curvature one in $\mathbb{H}^3$, special Lagrangian submanifolds.

Since the principal results of this book are on higher dimensional minimal submanifolds in various ambient manifolds, our exposition is based on properties of immersed submanifolds in a Riemannian manifold. We will make frequent use of well-known facts from Riemannian geometry; for the detailed background material we refer readers to standard textbooks.

This text is based on lectures that I presented a number of times at the Institute of Mathematics, Fudan University and other institutions in China.

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