

Contents

Preface and Acknowledgments	xi
Notations	xv
0 Lie Groups and Lie Algebras; Introduction	1
0.1 Topological Groups	1
0.2 Lie Groups	6
0.3 Covering Maps and Groups	10
0.4 Group Actions and Homogeneous Spaces	15
0.5 Lie Algebras	25
1 Lie Groups	31
1.1 Elementary Properties of a Lie Group	31
1.2 Taylor's Theorem and the Coefficients of $\exp X \exp Y$. .	39
1.3 Correspondence between Lie Subgroups and Subalgebras	45
1.4 The Functorial Relationship	48
1.5 The Topology of Compact Classical Groups	60
1.6 The Iwasawa Decompositions for $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$	67
1.7 The Baker-Campbell-Hausdorff Formula	69
2 Haar Measure and its Applications	89
2.1 Haar Measure on a Locally Compact Group	89
2.2 Properties of the Modular Function	100
2.3 Invariant Measures on Homogeneous Spaces	101
2.4 Compact or Finite Volume Quotients	106

2.5	Applications	112
2.6	Compact Linear Groups and Hilbert's 14 th Problem . .	121
3	Elements of the Theory of Lie Algebras	127
3.1	Basics of Lie Algebras	127
3.1.1	Ideals and Related Concepts	127
3.1.2	Semisimple Lie Algebras	138
3.1.3	Complete Lie Algebras	139
3.1.4	Lie Algebra Representations	140
3.1.5	The Irreducible Representations of $\mathfrak{sl}(2, k)$	142
3.1.6	Invariant Forms	145
3.1.7	Complex, Real and Rational Lie Algebras	147
3.2	Engel and Lie's Theorems	150
3.2.1	Engel's Theorem	150
3.2.2	Lie's Theorem	153
3.3	Cartan's Criterion and Semisimple Lie Algebras	157
3.3.1	Some Algebra	157
3.3.2	Cartan's Solvability Criterion	162
3.3.3	Explicit Computations of Killing Form	166
3.3.4	Further Results on Jordan Decomposition	170
3.4	Weyl's Theorem on Complete Reducibility	173
3.5	Levi-Malcev Decomposition	179
3.6	Reductive Lie Algebras	187
3.7	The Jacobson-Morozov Theorem	192
3.8	Low Dimensional Lie Algebras over \mathbb{R} and \mathbb{C}	198
3.9	Real Lie Algebras of Compact Type	202
4	The Structure of Compact Connected Lie Groups	207
4.1	Introduction	207
4.2	Maximal Tori in Compact Lie Groups	208
4.3	Maximal Tori in Compact Connected Lie Groups	210
4.4	The Weyl Group	217
4.5	What Goes Wrong If G is Not Compact	221

5	Representations of Compact Lie Groups	223
5.1	Introduction	224
5.2	The Schur Orthogonality Relations	226
5.3	Compact Integral Operators on a Hilbert Space	228
5.4	The Peter-Weyl Theorem and its Consequences	234
5.5	Characters and Central Functions	243
5.6	Induced Representations	250
5.7	Some Consequences of Frobenius Reciprocity	255
6	Symmetric Spaces of Non-compact Type	261
6.1	Introduction	261
6.2	The Polar Decomposition	264
6.3	The Cartan Decomposition	267
6.4	The Case of Hyperbolic Space and the Lorentz Group	274
6.5	The G -invariant Metric Geometry of P	278
6.6	The Conjugacy of Maximal Compact Subgroups	289
6.7	The Rank and Two-Point Homogeneous Spaces	294
6.8	The Disk Model for Spaces of Rank 1	299
6.9	Exponentiality of Certain Rank 1 Groups	304
7	Semisimple Lie Algebras and Lie Groups	313
7.1	Root and Weight Space Decompositions	313
7.2	Cartan Subalgebras	316
7.3	Roots of Complex Semisimple Lie Algebras	323
7.4	Real Forms of Complex Semisimple Lie Algebras	337
7.5	The Iwasawa Decomposition	343
8	Lattices in Lie Groups	355
8.1	Lattices in Euclidean Space	355
8.2	$GL(n, \mathbb{R})/GL(n, \mathbb{Z})$ and $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$	360
8.3	Lattices in More General Groups	371
8.4	Fundamental Domains	374
9	Density Results for Cofinite Volume Subgroups	377
9.1	Introduction	377
9.2	A Density Theorem for Cofinite Volume Subgroups	379

9.3	Consequences and Extensions of the Density Theorem .	389
A	Vector Fields	397
B	The Kronecker Approximation Theorem	403
C	Properly Discontinuous Actions	407
D	The Analyticity of Smooth Lie Groups	411
	Bibliography	413
	Index	421