Chapter 1

Introduction

In the last third of the 20th century efficient methods in automatic theorem proving in geometry were developed. By means of this theory hundreds of non-trivial theorems have been proved and even discovered. A correspondence between varieties and ideals gives us the ability to solve a given problem classically by means of the properties of geometric objects or by computer algebra methods based on the theory of automated theorem proving.

In the seminar which I led in the last years at the University of South Bohemia we solved on computers a number of problems of elementary geometry by the theory of automatic theorem proving. Students who attended this optional seminar were mostly at their 4th year of study, i.e., they had basic knowledge of geometry. Both by computers and in a classical way we investigated a number of tasks — the formula of Heron for the area of a triangle and its generalization — the formula of Brahmagupta for the area of an inscribed quadrilateral in terms of its lengths of sides, the formula of Staudt, Simson–Wallace theorem and its generalization, Napoleon's theorem and further similar problems.

A classical (or synthetic) method gives a better insight into the given geometrical situation which also enables a better understanding of a problem and shows the beauty of geometry. On the other side by computer algebra we can solve complex problems which are difficult to solve by classical approach. Using computer algebra we can carry out automatic proving theorems, automatic derivation and automatic discovery of new theorems. Whereas under automatic derivation we understand finding a statement which follows from the given assumptions, automatic discovery means searching for additional conditions which are necessary to add to the given assumptions so that the statement, which is not in general true, becomes valid. Automatic derivation is sometimes considered as a special case of automatic discovery. By computers we are also able to make constructions which are difficult to construct by the ruler and compass, etc.

First we give a brief overview of the theory of automatic theorem proving. To those who are interested in deeper study I recommend the book D. Cox, J. Little, D. O'Shea: Ideals, Varieties, and Algorithms [24] and D. Wang: Elimination Practice. Software Tools and Applications [136]. See also X. S. Gao, D. Wang (eds.): Mathematics Mechanization and Applications [43], where the overview of mechanizing mathematical activities, such as calculation, reasoning and discovery is given.

This book can be useful for those who are interested in geometry and proving theorems. In a few chapters which follow after the exposition of necessary theory on automatic theorem proving, both well-known and less known problems are solved — I called them stories — with the author's effort to describe their solution from the beginning to present time.

The first story "Generalization of the formula of Heron" in Chapter 3 deals with a generalization of Heron's and Brahmagupta's formulas for the area of a triangle and an inscribed quadrilateral in terms of the lengths of sides on an inscribed pentagon. This problem, which was solved in 1994 by D. P. Robbins [112], is investigated in this book by elimination of variables.

In the second story "Simson–Wallace Theorem" (Chapter 4) this well-known theorem of planar geometry is generalized into three dimensional space. First known generalizations of the Simson–Wallace theorem in a plane, from which particularly the Guzmán's generalization deserves attention, are demonstrated. Furthermore two spatial analogies of this theorem, which lead to the cubic surfaces with interesting properties, are shown.

Various generalizations of Ceva's, Menelaus' and Euler's theorems in a plane and space are studied in the part "Transversals in a polygon" (Chapter 5). In it, the power of the computer approach which searches for new formulas, and the power of the traditional yet possibly neglected area method (area principle), are shown.

The Napoleon's theorem and its generalization — Petr's theorem — and their planar and spatial analogies are given in the story "Petr–Douglas–Neumann theorem" (Chapter 6). As the name says, this theorem is connected with the well-known Czech mathematician K. Petr, a professor of the Charles University in Prague, who first published the theorem in 1905. Several special cases of the PDN theorem are solved — Thébault's theorem, Finsler–Hadwiger theorem, theorem of Finney, Van Aubel's theorem etc. Kiepert hyperbola, which is closely connected with Napoleon's theorem, is investigated as a locus of points of a given property.

In "Geometric inequalities" (Chapter 7), the inequality between lengths of sides and diagonals of a polygon in a plane and space is gradually generalized. The base is the well-known parallelogram law which played an important role in the thirties of the last century when it was shown that Banach space, in which parallelogram law holds, is the Hilbert space. Further the inequality between the sum of squares of sides of a polygon and its area is investigated. Then the Neuberg–Pedoe inequality for two triangles and the discrete case of Wirtinger's inequality are explored. The isoperimetric inequality for polygons concludes this family of inequalities. All these inequalities are proved using the elimination of variables and the sum of squares decomposition of polynomials.

Another method of proving inequalities using computer is shown on the well-known Euler's inequality between the radii of the circumcircle and the incircle of a triangle.

The next story "Regular polygons" (Chapter 8) is devoted to problems connected with regular polygons. Although it seems that all important things about regular polygons have been said, this is not the case. In 1969 two chemists visited the well-known mathematician B. L. Van der Waerden and stated that according to their investigations an equilateral and equiangular pentagon in a space must necessarily be planar. During a short period Van der Waerden and a number of further mathematicians proved that a regular polygon with an odd number of vertices always has even dimension. It is proved that regular pentagons and heptagons span spaces of even dimension.

Exploring properties of regular polygons in E^d, we might find ourselves in a situation where it is necessary to show that the conclusion polynomial belongs to the radical $\sqrt{I}$ of an ideal I. It is shown that in this case the respective ideal I is a proper subset of the radical $\sqrt{I}$ and we have to apply the stronger criterion, i.e. to show that c belongs to the radical $\sqrt{I}$.

In "Miscellaneous" (Chapter 9) a few problems of various parts of geometry which seem to be of interest are investigated. Especially the first two parts — "Non-elementary constructions" and "Loci of points of given properties" — are important both from the geometry and algebra point of view. Whereas loci of points belong to frequent issues of the theory of elimination, see [136], non-elementary constructions are not often mentioned in computer algebra topics. By means of a computer we are able to solve even such problems which are (Euclidean) unsolvable by a ruler and compass and were taboo in the past. Further the theorem of Viviani and the theorem of Gauss which is also known as the theorem on a complete quadrilateral are investigated. The theorem of Viviani is generalized using the method of automatic discovery based on the extension of the conclusion variety. Proving the theorem of Gauss automatically and classically we would realize that sometimes one method seems to be simpler than the other one.

At the end of this chapter is a list of the most common problems that students might encounter in the course of the seminar on automatic theorem proving at the university.

Classical solutions to all problems are mostly given. If a classical solution is missing it is likely caused by the fact that the author does not know it.

Computations were done in computer algebra system CoCoA[1]. Sometimes the programs Singular and Maple were used. Given computations can be carried out with another common mathematical software using elimination of variables based on Gröbner bases computation as well (Mathematica, Derive, ...).[2]

Figures are drawn in dynamic geometry software Cabri II Plus. All computations were done on a computer Intel Pentium 2.00GHz/1536MB RAM.

[1] Software CoCoA is freely distributed at the address `http://cocoa.dima.unige.it`

[2] The book [136] contains the application module GEOTHER working under Maple, which provides an environment for handling and proving theorems in geometry automatically.