## Chapter 1

# Sequences and Limits

- Let $\{a_n\}$ be a sequence of real or complex numbers. A necessary and sufficient condition for the sequence to converge is that for any $\epsilon > 0$ there exists an integer $N > 0$ such that

$$|a_p - a_q| < \epsilon$$

holds for all integers $p$ and $q$ greater than $N$. This is called the *Cauchy criterion*.

- Any monotone bounded sequence is convergent.
- For any sequence $\{a_n\}$ the *inferior limit* and the *superior limit* are defined by the limits of monotone sequences

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf \{a_n, a_{n+1}, \ldots\}$$

and

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup \{a_n, a_{n+1}, \ldots\}$$

respectively. Note that the inferior and superior limits always exist if we adopt $\pm\infty$ as limits.

- A bounded sequence $\{a_n\}$ converges if and only if the inferior limit coincides with the superior limit.

**Problem 1.1**

*Prove that* $n\sin(2n!e\pi)$ *converges to* $2\pi$ *as* $n \to \infty$.

**Problem 1.2**

*Prove that the sequence*

$$\left(\frac{1}{n}\right)^n + \left(\frac{2}{n}\right)^n + \cdots + \left(\frac{n}{n}\right)^n$$

*converges to* $e/(e-1)$ *as* $n \to \infty$.

**Problem 1.3**

*Prove that the sequence*

$$e^{n/4} n^{-(n+1)/2} \left(1^1 \cdot 2^2 \cdots n^n\right)^{1/n}$$

*converges to* 1 *as* $n \to \infty$.

This was proposed by Cesàro (1888) and solved by Pólya (1911).

**Problem 1.4**

*Suppose that* $a_n$ *and* $b_n$ *converge to* $\alpha$ *and* $\beta$ *as* $n \to \infty$ *respectively. Show that the sequence*

$$\frac{a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0}{n}$$

*converges to* $\alpha\beta$ *as* $n \to \infty$.

**Problem 1.5**

*Suppose that* $\{a_n\}_{n \geq 0}$ *is a non-negative sequence satisfying*

$$a_{m+n} \leq a_m + a_n + C$$

*for all positive integers* $m, n$ *and some non-negative constant* $C$. *Show that* $a_n/n$ *converges as* $n \to \infty$.

This is essentially due to Fekete (1923). In various places we encounter this useful lemma in deducing the existence of limits.

**Problem 1.6**

*For any positive sequence* $\{a_n\}_{n\geq 1}$ *show that*

$$\left(\frac{a_1 + a_{n+1}}{a_n}\right)^n > e$$

*for infinitely many n's, where e is base of the natural logarithm. Prove moreover that the constant e on the right-hand side cannot in general be replaced by any larger number.*

**Problem 1.7**

*For any* $0 < \theta < \pi$ *and any positive integer n show the inequality*

$$\sin\theta + \frac{\sin 2\theta}{2} + \cdots + \frac{\sin n\theta}{n} > 0.$$

This was conjectured by Fejér and proved by Jackson (1911) and by Gronwall (1912) independently. Landau (1934) gave a shorter (maybe the shortest) elegant proof. See also **Problem 5.9**. Note that

$$\sum_{n=1}^{\infty} \frac{\sin n\theta}{n} = \frac{\pi - \theta}{2}$$

for $0 < \theta < 2\pi$, which is shown in **Solution 7.10**.

**Problem 1.8**

*For any real number* $\theta$ *and any positive integer n show the inequality*

$$\frac{\cos\theta}{2} + \frac{\cos 2\theta}{3} + \cdots + \frac{\cos n\theta}{n+1} \geq -\frac{1}{2}.$$

This was shown by Rogosinski and Szegö (1928). Verblunsky (1945) gave another proof. Koumandos (2001) obtained the lower bound $-41/96$ for $n \geq 2$.

Note that

$$\sum_{n=0}^{\infty} \frac{\cos n\theta}{n+1} = \frac{\pi - \theta}{2}\sin\theta - \cos\theta \log\left(2\sin\frac{\theta}{2}\right)$$

for $0 < \theta < 2\pi$.

For the simpler cosine sum Young (1912) showed that

$$\sum_{n=1}^{m} \frac{\cos n\theta}{n} > -1$$

for any $\theta$ and positive integer $m \geq 2$. Brown and Koumandos (1997) improved this by replacing $-1$ by $-5/6$.

**Problem 1.9**

*Given a positive sequence $\{a_n\}_{n\geq 0}$ satisfying $\sqrt{a_1} \geq \sqrt{a_0} + 1$ and*

$$\left| a_{n+1} - \frac{a_n^2}{a_{n-1}} \right| \leq 1$$

*for any positive integer $n$, show that*

$$\frac{a_{n+1}}{a_n}$$

*converges as $n \to \infty$. Show moreover that $a_n \theta^{-n}$ converges as $n \to \infty$, where $\theta$ is the limit of the sequence $a_{n+1}/a_n$.*

This is due to Boyd (1969).

**Problem 1.10**

*Let $E$ be any bounded closed set in the complex plane containing an infinite number of points, and let $M_n$ be the maximum of $|V(x_1, ..., x_n)|$ as the points $x_1$, ..., $x_n$ run through the set $E$, where*

$$V(x_1, ..., x_n) = \prod_{1\leq i<j\leq n} (x_i - x_j)$$

*is the Vandermonde determinant. Show that $M_n^{2/(n(n-1))}$ converges as $n \to \infty$.*

This is due to Fekete (1923) and the limit

$$\tau(E) = \lim_{n\to\infty} M_n^{2/(n(n-1))}$$

is called the *transfinite diameter* of $E$. See **Problem 15.9**.

# Solutions for Chapter 1

**Solution 1.1**

Let $r_n$ and $\epsilon_n$ be the integral and fractional parts of the number $n!e$ respectively. Using the expansion

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots ,$$

we have

$$\begin{cases} r_n = n!\left(1 + \dfrac{1}{1!} + \dfrac{1}{2!} + \cdots + \dfrac{1}{n!}\right) \\[2ex] \epsilon_n = \dfrac{1}{n+1} + \dfrac{1}{(n+1)(n+2)} + \cdots , \end{cases}$$

since

$$\frac{1}{n+1} < \epsilon_n < \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots = \frac{1}{n}.$$

Thus $\sin(2n!e\pi) = \sin(2\pi\epsilon_n)$. Note that this implies the irrationality of $e$.

Since $n\epsilon_n$ converges to 1 as $n \to \infty$, we have

$$\lim_{n\to\infty} n\sin(2\pi\epsilon_n) = \lim_{n\to\infty} \frac{\sin(2\pi\epsilon_n)}{\epsilon_n} = 2\pi.$$

Hence $n\sin(2n!e\pi)$ converges to $2\pi$ as $n \to \infty$. □

**Remark.** More precisely one gets

$$\epsilon_n = \frac{1}{n} - \frac{1}{n^3} + O\left(\frac{1}{n^4}\right);$$

hence we have

$$\begin{aligned} n\sin(2n!e\pi) &= 2\pi n\epsilon_n + \frac{4\pi^3}{3}n\epsilon_n^3 + O(n\epsilon_n^5) \\ &= 2\pi + \frac{2\pi(2\pi^2 - 3)}{3n^2} + O\left(\frac{1}{n^3}\right) \end{aligned}$$

as $n \to \infty$.

**Solution 1.2**

Let $\{d_n\}$ be any monotone increasing sequence of positive integers diverging to $\infty$ and satisfying $d_n < n$ for $n > 1$. We divide the sum into two parts as follows.

$$\begin{cases} a_n = \left(\dfrac{1}{n}\right)^n + \left(\dfrac{2}{n}\right)^n + \cdots + \left(\dfrac{n-1-d_n}{n}\right)^n, \\[2ex] b_n = \left(\dfrac{n-d_n}{n}\right)^n + \cdots + \left(\dfrac{n}{n}\right)^n. \end{cases}$$

First the sum $a_n$ is roughly estimated above by

$$\frac{1}{n^n}\int_0^{n-d_n} x^n\,dx = \frac{(n-d_n)^{n+1}}{(n+1)n^n} < \left(1 - \frac{d_n}{n}\right)^n.$$

Now using the inequality $\log(1-x) + x < 0$ valid for $0 < x < 1$ we obtain

$$0 < a_n < e^{n\log(1-d_n/n)} < e^{-d_n},$$

which converges to 0 as $n \to \infty$.

Next by using Taylor's formula for $\log(1-x)$ we can take a positive constant $c_1$ such that the inequality

$$|\log(1-x) + x| \le c_1 x^2$$

holds for any $|x| \le 1/2$. Thus for any integer $n$ satisfying $d_n/n \le 1/2$ we get

$$\left| n\log\left(1 - \frac{k}{n}\right) + k \right| \le \frac{c_1 k^2}{n} \le \frac{c_1 d_n^2}{n}$$

for $0 \le k \le d_n$. Suppose further that $d_n^2/n$ converges to 0 as $n \to \infty$. For example $d_n = [n^{1/3}]$ satisfies all the conditions imposed above. Next take a positive constant $c_2$ satisfying

$$|e^x - 1| \le c_2|x|$$

for any $|x| \le 1$. Since $c_1 d_n^2/n \le 1$ for all sufficiently large $n$, we have

$$\begin{aligned} \left| e^k\left(1 - \frac{k}{n}\right)^n - 1 \right| &= \left| e^{n\log(1-k/n)+k} - 1 \right| \\ &\le \frac{c_1 c_2 d_n^2}{n}. \end{aligned}$$

Dividing both sides by $e^k$ and summing from $k = 0$ to $d_n$, we get

$$\sum_{k=0}^{d_n} \left| \left(1 - \frac{k}{n}\right)^n - e^{-k} \right| \le \frac{c_1 c_2 d_n^2}{n} \sum_{k=0}^{d_n} e^{-k}.$$

Hence

$$\left| b_n - \sum_{k=0}^{d_n} e^{-k} \right| < \frac{e c_1 c_2 d_n^2}{(e-1)n},$$

which implies

$$\left| b_n - \frac{e}{e-1} \right| \le \frac{e}{e-1} \left( \frac{c_1 c_2 d_n^2}{n} + e^{-d_n} \right).$$

Therefore $a_n + b_n$ converges to $e/(e-1)$ as $n \to \infty$. □

**SOLUTION 1.3**

One can easily verify that the function $f(x) = x \log x$ satisfies all the conditions stated in **PROBLEM 5.7**. Therefore the logarithm of the given sequence converges to

$$\frac{f(1) - f(0+)}{2} = 0,$$

hence the limit is 1. □

**SOLUTION 1.4**

Let $M$ be an upper bound of the two convergent sequences $|a_n|$ and $|b_n|$. For any $\epsilon > 0$ we can take a positive integer $N$ satisfying $|a_n - \alpha| < \epsilon$ and $|b_n - \beta| < \epsilon$ for all integers $n$ greater than $N$. If $n$ is greater than $N^2$, then

$$\begin{aligned} |a_k b_{n-k} - \alpha\beta| &\le |(a_k - \alpha) b_{n-k} + \alpha(b_{n-k} - \beta)| \\ &\le (M + |\alpha|)\epsilon \end{aligned}$$

for any integer $k$ in the interval $[\sqrt{n}, n - \sqrt{n}]$. Therefore

$$\begin{aligned} \left| \frac{1}{n} \sum_{k=0}^{n} a_k b_{n-k} - \alpha\beta \right| &\le \frac{1}{n} \sum_{\sqrt{n} \le k \le n - \sqrt{n}} |a_k b_{n-k} - \alpha\beta| \\ &\qquad + 2\left(|\alpha\beta| + M^2\right) \frac{[\sqrt{n}\,] + 1}{n} \\ &\le (M + |\alpha|)\epsilon + 2\left(|\alpha\beta| + M^2\right) \frac{\sqrt{n} + 1}{n}. \end{aligned}$$

We can take $n$ so large that the last expression is less than $(M + |\alpha| + 1)\epsilon$. □

**Solution 1.5**

For an arbitrary fixed positive integer $k$ we put $n = qk + r$ with $0 \le r < k$. Since $a_n = a_{qk+r} \le q(a_k + C) + a_r$, we have

$$\frac{a_n}{n} \le \frac{a_k + C}{k} + \frac{a_r}{n}.$$

Taking the limit as $n \to \infty$, we get

$$\limsup_{n\to\infty} \frac{a_n}{n} \le \frac{a_k + C}{k}.$$

The sequence $a_n/n$ is therefore bounded. Since $k$ is arbitrary, we may conclude that

$$\limsup_{n\to\infty} \frac{a_n}{n} \le \liminf_{k\to\infty} \frac{a_k}{k},$$

which means the convergence of $a_n/n$. □

**Solution 1.6**

Without loss of generality we may put $a_1 = 1$. Suppose, contrary to the conclusion, that there is an integer $N$ satisfying

$$\left(\frac{1 + a_{n+1}}{a_n}\right)^n \le e$$

for all $n \ge N$. Put

$$s_{j,k} = \exp\left(\frac{1}{j} + \cdots + \frac{1}{k}\right)$$

for any integers $j \le k$. Since $0 < a_{n+1} \le e^{1/n} a_n - 1$, we get successively

$$\begin{cases} 0 < a_{n+1} \le s_{n,n}\, a_n - 1, \\ 0 < a_{n+2} \le s_{n,n+1}\, a_n - s_{n+1,n+1} - 1, \\ \qquad\qquad \vdots \\ 0 < a_{n+k+1} \le s_{n,n+k}\, a_n - s_{n+1,n+k} - \cdots - s_{n+k,n+k} - 1 \end{cases}$$

for any non-negative integer $k$. Hence it follows that

$$a_n > \frac{1}{s_{n,n}} + \frac{1}{s_{n,n+1}} + \cdots + \frac{1}{s_{n,n+k}}.$$

On the other hand, using the inequality

$$\frac{1}{s_{n,n+j}} > \exp\left(-\int_{n-1}^{n+j}\frac{dx}{x}\right) = \frac{n-1}{n+j},$$

we get

$$a_n > \sum_{j=0}^{k}\frac{n-1}{n+j},$$

which is a contradiction, since the right-hand side diverges to $\infty$ as $k \to \infty$.

To see that the bound $e$ cannot be replaced by any larger number, consider the case $a_n = n\log n$ for $n \geq 2$. Then

$$\begin{aligned}\left(\frac{a_1 + (n+1)\log(n+1)}{n\log n}\right)^n &= \exp\left(n\log\left(1+\frac{1}{n}+O\Big(\frac{1}{n\log n}\Big)\right)\right)\\ &= \exp\left(1+O\Big(\frac{1}{\log n}\Big)\right),\end{aligned}$$

which converges to $e$ as $n \to \infty$. □

**SOLUTION 1.7**

Denote by $s_n(\theta)$ the left-hand side of the inequality to be shown. Write $\theta$ for $2\vartheta$ for brevity. Since

$$\begin{aligned}s_n'(\theta) &= \Re\left(e^{i\theta} + e^{2i\theta} + \cdots + e^{ni\theta}\right)\\ &= \frac{\cos(n+1)\vartheta\sin n\vartheta}{\sin\vartheta},\end{aligned}$$

we obtain the candidates for extreme points of $s_n(\theta)$ on the interval $(0,\pi]$ by solving the equations $\cos(n+1)\vartheta = 0$ and $\sin n\vartheta = 0$, as follows:

$$\frac{\pi}{n+1},\ \frac{2\pi}{n},\ \frac{3\pi}{n+1},\ \frac{4\pi}{n},\ \ldots$$

where the last two candidates are $(n-1)\pi/(n+1)$ and $\pi$ if $n$ is even, and $(n-1)\pi/n$ and $n\pi/(n+1)$ if $n$ is odd. In any case $s_n'(\theta)$ vanishes at least at $n$ points in the interval $(0,\pi)$.

Since $s_n'(\theta)$ can be expressed as a polynomial in $\cos\theta$ of degree $n$ and $\cos\theta$ maps the interval $[0,\pi]$ onto $[-1,1]$ homeomorphically, this polynomial possesses at most $n$ real roots in $[-1,1]$. Therefore all these roots must be simple and give the actual extreme points of $s_n(\theta)$ except for $\theta = \pi$. Clearly $s_n(\theta)$ is positive in the right neighborhood of the origin, and the maximal and minimal points stand in line alternately from left to right. Thus $s_n(\theta)$ attains its minimal values at the points $2\ell\pi/n \in (0,\pi)$ when $n \geq 3$. In the cases $n = 1$ and $n = 2$, however, $s_n(\theta)$ has no minimal points in $(0,\pi)$.

Now we will show that $s_n(\theta)$ is positive on the interval $(0,\pi)$ by induction on $n$. This is clear for $n = 1$ and $n = 2$ since $s_1(\theta) = \sin\theta$ and $s_2(\theta) = (1+\cos\theta)\sin\theta$. Suppose that $s_{n-1}(\theta) > 0$ for some $n \geq 3$. Then the minimal values of $s_n(\theta)$ are certainly attained at some points $2\ell\pi/n$ in $(0,\pi)$, whose values are

$$\begin{aligned} s_n\left(\frac{2\ell\pi}{n}\right) &= s_{n-1}\left(\frac{2\ell\pi}{n}\right) + \frac{\sin 2\ell\pi}{n} \\ &= s_{n-1}\left(\frac{2\ell\pi}{n}\right) > 0. \end{aligned}$$

Therefore $s_n(\theta) > 0$ on the interval $(0,\pi)$. □

**Remark.** Landau (1934) gave the following elegant shorter proof using mathematical induction on $n$. Suppose that $s_{n-1}(\theta) > 0$ on $(0,\pi)$. If $s_n$ attains the non-positive minimum at some point, say $\theta^*$, then $s_n'(\theta^*) = 0$ implies

$$\sin\left(n + \frac{1}{2}\right)\theta^* = \sin\frac{\theta^*}{2}$$

and hence

$$\cos\left(n + \frac{1}{2}\right)\theta^* = \pm\cos\frac{\theta^*}{2}.$$

Since

$$\begin{aligned} \sin n\theta^* &= \sin\left(n + \frac{1}{2}\right)\theta^* \cos\frac{\theta^*}{2} - \cos\left(n + \frac{1}{2}\right)\theta^* \sin\frac{\theta^*}{2} \\ &= \sin\frac{\theta^*}{2}\cos\frac{\theta^*}{2} \pm \cos\frac{\theta^*}{2}\sin\frac{\theta^*}{2}, \end{aligned}$$

being equal either to 0 or $\sin\theta^* \geq 0$ according to the sign. We are led to a contradiction.

## Solution 1.8

The proof is substantially based on Verblunsky (1945). Write $\theta$ for $2\vartheta$ for brevity. Let $c_n(\vartheta)$ be the left-hand side of the inequality to be shown. It suffices to confine ourselves to the interval $[0,\pi/2]$. Clearly $c_1(\vartheta) = \cos\vartheta/2 \geq -1/2$ and

$$c_2(\vartheta) = \frac{2}{3}\cos^2\vartheta + \frac{1}{2}\cos\vartheta - \frac{1}{3} \geq -\frac{41}{96},$$

and we assume that $n \geq 3$. Note that

$$\cos n\theta = \frac{\sin(2n+1)\vartheta - \sin(2n-1)\vartheta}{2\sin\vartheta}$$
$$= \frac{\sin^2(n+1)\vartheta - 2\sin^2 n\vartheta + \sin^2(n-1)\vartheta}{2\sin^2\vartheta},$$

whose numerator is the second difference of the positive sequence $\{\sin^2 n\vartheta\}$. Using this formula we get

$$c_n(\vartheta) = \frac{1}{2\sin^2\vartheta}\sum_{k=1}^{n}\frac{\sin^2(k+1)\vartheta - 2\sin^2 k\vartheta + \sin^2(k-1)\vartheta}{k+1},$$

which can be written as

$$\frac{1}{2\sin^2\vartheta}\Bigg(-\frac{2\sin^2\vartheta}{3} + \frac{\sin^2 2\vartheta}{12} + \cdots + \frac{2\sin^2(n-1)\vartheta}{n(n^2-1)} - \frac{(n-1)\sin^2 n\vartheta}{n(n+1)} + \frac{\sin^2(n+1)\vartheta}{n+1}\Bigg).$$

Hence we obtain

$$c_n(\vartheta) \geq -\frac{1}{3} + \frac{\cos^2\vartheta}{6} + \frac{\sin^2(n+1)\vartheta - \sin^2 n\vartheta}{2(n+1)\sin^2\vartheta}$$
$$= -\frac{1}{6} - \frac{\sin^2\vartheta}{6} + \frac{\sin(2n+1)\vartheta}{2(n+1)\sin\vartheta}.$$

For any $\vartheta$ satisfying $\sin(2n+1)\vartheta \geq 0$ we obviously have $c_n(\vartheta) \geq -1/3$. Moreover if $\vartheta$ belongs to the interval $(3\pi/(2n+1), \pi/2)$, then using Jordan's inequality $\sin\vartheta \geq 2\vartheta/\pi$,

$$c_n(\vartheta) \geq -\frac{1}{3} - \frac{1}{2(n+1)\sin(3\pi/(2n+1))}$$
$$\geq -\frac{1}{3} - \frac{2n+1}{12(n+1)} > -\frac{1}{2}.$$

Thus it suffices to consider the interval $[\pi/(2n+1), 2\pi/(2n+1)]$.

In general, we consider an interval of the form

$$\left[\frac{\alpha\pi}{2n+1}, \frac{\beta\pi}{2n+1}\right].$$

For any $\vartheta$ satisfying $\sin(2n+1)\vartheta \leq c$ on this interval it follows that

$$c_n(\vartheta) \geq -\frac{1}{6} - \frac{\sin^2\vartheta}{6} - \frac{c}{2(n+1)\sin\vartheta}.$$

Now the right-hand side can be written as $-1/6-\varphi(\sin\vartheta)$, where $\varphi(x)$ is a concave function; hence, the maximum of $\varphi$ is attained at an end point of that interval. By using

$$\alpha\pi\sin\vartheta \geq 7\vartheta\sin\frac{\alpha\pi}{7},$$

we get

$$\begin{aligned}\varphi\left(\sin\frac{\alpha\pi}{2n+1}\right) &= \frac{1}{6}\sin^2\frac{\alpha\pi}{2n+1} + \frac{c}{2(n+1)\sin(\alpha\pi/(2n+1))}\\ &\leq \frac{(\alpha\pi)^2}{6(2n+1)^2} + \frac{c}{2(n+1)}\cdot\frac{2n+1}{7\sin(\alpha\pi/7)}.\end{aligned}$$

Since $n \geq 3$, the last expression is less than

$$\frac{(\alpha\pi)^2}{294} + \frac{c}{7\sin(\alpha\pi/7)}.$$

Similarly we get an estimate for another end point.

For $\alpha = 1$ and $\beta = 4/3$ we can take $c = \sqrt{3}/2$ so that the value of $\varphi$ at the corresponding end point is less than 0.319 and 0.28 respectively. Similarly for $\alpha = 4/3$ and $\beta = 2$ we can take $c = 1$ so that the value of $\varphi$ is less than 0.314 and 0.318 respectively. Therefore the maximum of $\varphi$ on the interval $[\pi/(2n+1), 2\pi/(2n+1)]$ is less than $1/3$, which implies that $c_n(\vartheta) > -1/2$. □

**Solution 1.9**

We first show that

$$\frac{a_{n+1}}{a_n} > 1 + \frac{1}{\sqrt{a_0}} \tag{1.1}$$

by induction on $n$. When $n = 0$ this holds by the assumption. Put $\alpha = 1 + 1/\sqrt{a_0}$ for brevity. Suppose that (1.1) holds for $n \leq m$. We then have $a_k > \alpha^k a_0$ for $1 \leq k \leq m+1$. Thus

$$\left|\frac{a_{m+2}}{a_{m+1}} - \frac{a_1}{a_0}\right| \leq \sum_{k=1}^{m+1}\left|\frac{a_{k+1}}{a_k} - \frac{a_k}{a_{k-1}}\right| \leq \sum_{k=1}^{m+1}\frac{1}{a_k},$$

which is less than

$$\frac{1}{a_0}\sum_{k=1}^{m+1}\alpha^{-k} < \frac{1}{a_0(\alpha-1)} = \frac{1}{\sqrt{a_0}}.$$

Therefore

$$\frac{a_{m+2}}{a_{m+1}} > \frac{a_1}{a_0} - \frac{1}{\sqrt{a_0}} > 1 + \frac{1}{\sqrt{a_0}};$$

thus (1. 1) holds also for $n = m + 1$.

Let $p > q$ be any positive integers. In the same way,

$$\left| \frac{a_{p+1}}{a_p} - \frac{a_{q+1}}{a_q} \right| \le \sum_{k=q+1}^{p} \left| \frac{a_{k+1}}{a_k} - \frac{a_k}{a_{k-1}} \right| \le \sum_{k=q+1}^{p} \frac{1}{a_k},$$

which is less than

$$\frac{1}{a_q} \sum_{k=1}^{p-q} \frac{1}{\alpha^k} < \frac{\sqrt{a_0}}{a_q}.$$

This means that the sequence $\{a_{n+1}/a_n\}$ satisfies the Cauchy criterion since $a_q$ diverges to $\infty$ as $q \to \infty$. Letting $p \to \infty$ in the above inequalities, we get

$$\left| \frac{a_{q+1}}{a_q} - \theta \right| \le \frac{\sqrt{a_0}}{a_q}.$$

Multiplying both sides by $a_q/\theta^{q+1}$, we have

$$\left| \frac{a_{q+1}}{\theta^{q+1}} - \frac{a_q}{\theta^q} \right| \le \frac{\sqrt{a_0}}{\theta^{q+1}},$$

which shows that the sequence $\{a_n/\theta^n\}$ also satisfies the Cauchy criterion. □

**Solution 1. 10**

Let $\xi_1$, ..., $\xi_{n+1}$ be the points at which $|V(x_1, ..., x_{n+1})|$ attains its maximum $M_{n+1}$. Since

$$\frac{V(\xi_1, ..., \xi_{n+1})}{V(\xi_1, ..., \xi_n)} = (\xi_1 - \xi_{n+1}) \cdots (\xi_n - \xi_{n+1}),$$

we have

$$\frac{M_{n+1}}{M_n} \le |\xi_1 - \xi_{n+1}| \cdots |\xi_n - \xi_{n+1}|.$$

Applying the same argument to each point $\xi_1$, ..., $\xi_n$, we get $n + 1$ similar inequalities whose product gives

$$\left( \frac{M_{n+1}}{M_n} \right)^{n+1} \le \prod_{i \ne j} |\xi_i - \xi_j| = M_{n+1}^2.$$

Hence the sequence $M_n^{2/(n(n-1))}$ is monotone decreasing. □