Preface

This book is a continuation and development of the author's book (see [86]33)), and mainly deals with various boundary value problems for equations and systems of mixed (elliptic-hyperbolic) type with parabolic degeneracy. For this the corresponding boundary value problems for elliptic and hyperbolic complex equations of first and second orders are firstly considered, in which some representations and a priori estimates of solutions for the above problems are given, and the existence and uniqueness of solutions of these problems are proved. In the whole book we apply the complex analytic method in details.

In Chapters I and II, by using some new methods, we mainly investigate some discontinuous boundary value problems for some classes of elliptic complex equations of first and second orders with smooth and nonsmooth parabolic degenerate lines, which include the discontinuous Riemann-Hilbert boundary value problem, the mixed boundary value problem and discontinuous oblique derivative boundary value problem. In which we first reduce the degenerate elliptic equations or systems to some complex equations with singular coefficients, after this it is not very difficult to obtain a priori estimates of solutions for the above boundary value problems, and then the existence and uniqueness of solutions of these problems can be proved. As an application of the above results, we discuss a boundary value problem in axisymmetric filtration with homogeneous medium.

In Chapter III, on the basis of notations of hyperbolic numbers and hyperbolic complex functions, the hyperbolic systems of first order equations and hyperbolic equations of second order with some conditions can be reduced to complex forms. Moreover, several boundary value problems, mainly the Riemann-Hilbert boundary value problem, oblique derivative boundary value problem for first and second orders hyperbolic complex equations with parabolic degeneracy are discussed, which includes the Dirichlet boundary value problem as a special case. In addition, we discuss the Cauchy problem for second order hyperbolic equations with degenerate rank 0.

In Chapter IV, by using the notations of complex numbers in elliptic domains and hyperbolic numbers in hyperbolic domains, we mainly introduce the Riemann-Hilbert boundary value problem for first order linear and quasilinear complex equations of mixed type in special domains and general domains, the results obtained in which are the preparation for latter chapters.

For the classical gas dynamical equation of mixed type due to S. A. Chaplygin [17], the first really deep results were published by F. G. Tricomi [78]1). In Chapters V and VI, on the basis of the results obtained in Chapters I–IV, we consider the Tricomi boundary value problem, oblique derivative problem for second order linear and quasilinear complex equations of mixed type with parabolic degeneracy in several domains including general domains and multiply connected domains. We mention that in the books [12]1),3) and [74], the authors investigated the Tricomi problem for the Chaplygin equation: $K(y)u_{xx}+u_{yy}=0$ and special second order equations of mixed type with parabolic degeneracy in some standard domains and general domains by using the methods of integral equations and energy integral, but the methods are not simple. In the present book, we apply the uniqueness and existence of solutions of discontinuous boundary value problems for elliptic, hyperbolic complex equations and other methods to obtain the solvability results of several discontinuous oblique derivative problems for second order equations of mixed type, which include the Tricomi problem as a special case. Besides, we also discuss the Frankl problem and the exterior Tricomi problem for general equations of mixed type with parabolic degeneracy.

There are two characteristics of this book: one is that elliptic, hyperbolic and mixed complex equations are included in several forms and the quasilinear case, and boundary value conditions are almost considered in the discontinuous Riemann-Hilbert problem and oblique derivative problem, especially multiply connected domains are considered. Another one is that several complex methods are used to investigate various problems about complex equations of elliptic, hyperbolic and mixed type, for example the complex functions in elliptic domains and the hyperbolic complex functions in hyperbolic domains are used, and in general we first discuss the corresponding problems for first order complex equations, and then the problems for second order equations can be solved. The above method is different from the methods used by other authors. We mention that some boundary value problems in gas dynamics can be handled by using the results as stated in this book.

The great majority of the contents originates in investigations of the author and cooperative colleagues, and many results are published here for the first time, in which many open problems are solved, for instance the Tricomi problem for second order equations of mixed type in special and general multiply connected domains posed by L. Bers in [9]2), and the existence, regularity of solutions of some boundary value problems for second order equations of mixed type with smooth and nonsmooth degenerate lines posed by J. M. Rassias in [71]2). After reading the volume, it can be seen that many questions remain for further investigations.

In order to conveniently understand the purpose of the book, we give a sketchy introduction about the idea of the book. It is known that the general second order linear equation of mixed type is as follows

$$K(y)u_{xx}+u_{yy}+a(x,y)u_x+b(x,y)u_y+c(x,y)u+d(x,y)=0 \text{ in } D, \qquad (1)$$

especially

$$K(y)u_{xx}+u_{yy}=0 \text{ in } D \qquad (2)$$

is the so-called famous Chaplygin equation in gas dynamics, where $K(y)$ possesses the derivative $K'(y)$ and $yK(y)>0$ on $y\neq 0$, $K(0)=0$, and D is a bounded domain including the domain D^+ in the upper-half plane, the domain D^- in the lower-half plane and the line γ on the real axis, obviously equation (1) in D^+ is the elliptic type and in D^- is the hyperbolic type, and γ is a parabolic degenerate line. From [9]2) and [71]2), we can see the mechanical background of equations (1) and (2). The so-called Tricomi problem is to find a solution $u(z)$ of equation (1) or (2) satisfying the boundary conditions

$$u(z)=\phi(z) \text{ on } \Gamma,\ u(z)=\psi(z) \text{ on } L_1 \text{ or } L_2, \qquad (3)$$

where Γ is the partial boundary of D^+ in the upper-half plane and L_1, L_2 are two characteristics in the lower-half plane, here $L_1\cup L_2\cup\gamma$ is the boundary of D^-, and $z_0=x_0+jy_0$ is the intersection point of L_1 and L_2. The above Tricomi problem can be divided into two boundary value problems, namely the mixed boundary value problem

$$u(z)=\phi(z) \text{ on } \Gamma,\ u_y=r(x) \text{ on } \gamma, \qquad (4)$$

of equation (1) in D^+, and the mixed boundary value problem

$$u(z)=\psi(z) \text{ on } L_1 \text{ or } L_2,\ u_y=r(x) \text{ on } \gamma \qquad (5)$$

of equation (1) in D^-, where $r(x)$ on γ is an undetermined function, which can be determined by the boundary condition of Problem T of the mixed equations.

According to the mathematical view, the centrical subject of the Tricomi problem is to prove the existence and uniqueness of solutions for the Tricomi problem, and verify some regularity of solutions of the above problem. In recent half century, many mathematical authors investigated the problem about several equations of mixed type and obtained many interesting results (see [9]2), [12]1),3), [71], [74], [86]33) and so on). However the Tricomi problem of equation (1) has not been completely solved, and the obtained results almost require some unnecessary conditions, besides for instance the Tricomi problem for second order equations of mixed type in multiply connected domains posed by L. Bers in the book [9]2), i.e. Tricomi-Bers problem, and the existence, regularity of solutions of above problems for mixed equations with nonsmooth degenerate line in several domains posed by J. M. Rassias in the book [71]1), i.e. Tricomi-Frankl-Rassias problems have not been solved. The purpose of the book is just to introduce some new results about the above problems. As stated before we need first introduce the related results about degenerate elliptic equations and degenerate hyperbolic equations, and then the problems about equations of mixed type are discussed (see Sections 3 and 4, Chapter VI).

Our method of handling mixed equations is the complex analytic method, which is different to the methods of other authors, i.e. through the transformation of functions: $W(z) = [H(y)u_x - iu_y]/2$ in D^+ and $W(z) = [H(y)u_x - ju_y]/2$ in D^-, here $H(y) = \sqrt{|K(y)|}$, $Y = G(y) = \int_0^y H(t)dt$, i and j are the imaginary unit and hyperbolic imaginary unit with the conditions $i^2 = -1$ and $j^2 = 1$ respectively, and denote $Z = x + iY$ in $\overline{D^+}$ and $Z = x + jY$ in $\overline{D^-}$, the equation (1) can be reduced to the complex equation of first order with singular coefficients

$$W_{\overline{Z}} = A_1(Z)W(Z) + A_2(Z)\overline{W(Z)} + A_3(Z)u(Z) + A_4(Z) \text{ in } D_Z, \qquad (6)$$

where D_Z is the image domain of D with respect to the mapping $Z = Z(z) = x+iY = x+iG(y)$ in D^+ and $Z = Z(z) = x+jY = x+jG(y)$ in D^-. Moreover we find the directional derivatives of boundary condition (3) or (4), (5) according to the parameter of arc length of the boundaries Γ and L_1 or L_2, then (4), (5) can be written in the complex form

$$\text{Re}[\overline{\lambda(Z)}W(Z)] = R(Z) \text{ on } \Gamma \cup L_1 \cup \gamma \text{ or } \Gamma \cup L_2 \cup \gamma, \qquad (7)$$

in which $\lambda(Z) = i$ or $-j$, $R(Z) = -r(z)/2$ on γ, this is a boundary condition of Riemann-Hilbert type, we can use the similar method as stated in [86]11),33) and [87]1) to solve the problem of equation (6), but we must give some important modifications, and then the Tricomi problem for equation (1) can be solved.

In this book, we mainly consider three classes of second order equations of mixed type with parabolic degeneracy, i.e. the equation (1),

$$K_1(y)u_{xx}+|K_2(x)|u_{yy}+a(x,y)u_x+b(x,y)u_y+c(x,y)u+d(x,y)=0 \text{ in } D, \quad (8)$$

and

$$K_1(y)u_{xx}+|K_2(y)|u_{yy}+a(x,y)u_x+b(x,y)u_y+c(x,y)u+d(x,y)=0 \text{ in } D, \quad (9)$$

in which $K_1(y)$, $K_2(y)$ satisfy the condition similar to that of $K(y)$ in (1), and $K_2(x)$ possesses the derivative $K_2'(x)$ and $xK_2(x) > 0$ on $x \neq 0$, $K_2(0)=0$. It is easy to see that in (8), the real axis and the imaginary axis are parabolic degenerate lines, hence the parabolic degenerate curve is not smooth near the origin point. Except the above linear equations, we also discuss some quasilinear or nonlinear elliptic, hyperbolic and mixed equations with parabolic degeneracy.

The more general boundary value problem is the oblique derivative boundary value problem, which is to find a continuous solution $u(z)$ of (1) in $\overline{D}$ satisfying the boundary conditions

$$\begin{gathered}\frac{1}{2}\frac{\partial u}{\partial l}=\frac{1}{H(y)}\mathrm{Re}[\overline{\lambda(z)}u_{\tilde{z}}]=\mathrm{Re}[\overline{\Lambda(z)}u_z]=r(z) \text{ on } \Gamma\cup L_1,\\ \frac{1}{H(y)}\mathrm{Im}[\overline{\lambda(z)}u_{\tilde{z}}]|_{z=z_0}=b_0, u(z_1)=b_1, u(z_2)=b_2,\end{gathered} \quad (10)$$

in which l is a given vector at every point $z \in \Gamma \cup L_1$, $W(z) = u_{\tilde{z}} = [H(y)u_x - iu_y]/2$, $\Lambda(z) = \cos(l,x) - i\cos(l,y), \lambda(z) = \mathrm{Re}\lambda(z) + i\mathrm{Im}\lambda(z)$, $\cos(l,n) \geq 0$ for $z \in \Gamma$, n is the outward normal vector of Γ, and $W(z) = u_{\tilde{z}} = [H(y)u_x - ju_y]/2$, $\lambda(z) = \mathrm{Re}\lambda(z) + j\mathrm{Im}\lambda(z)$ for $z \in L_1$, z_1, z_2 are two end points of γ, b_0, b_1, b_2 are real constants. We can give the well-posed formulation according to the index of boundary conditions in the elliptic domain. It is clear that the oblique derivative problem includes the Tricomi problem as a special case. In the book, we mainly handle the oblique derivative problem about elliptic equations, hyperbolic equations and mixed equations with parabolic degeneracy, which include the above three classes of equations. Except the above boundary value problems, we also introduce the discontinuous oblique derivative problem, exterior Tricomi-Rassias problem and the Frankl problem in Sections 3-5 of Chapter V. Besides, the Tricomi problem for second order equations of mixed type with parabolic degeneracy in general domains and multiply connected domains are discussed. This book has not been related to the Tricomi problem for equations in higher dimensional domains (see [71]2),6)), some problems on this hand remain to be further investigated.

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