Chapter 1

Formal Stochastic Differential Equations

The goal of this first chapter is to establish the Chen-Strichartz formula which, in a way, is a cornerstone of this book. This formula is universal and determines very precisely and explicitly the local structure of any stochastic flow. To derive this formula, it is quite convenient to work in an abstract and formal setting, in which we do not have to care about convergence questions.

The reader which is not so familiar with the theory of stochastic differential equations and vector fields is invited read the Appendices A and B which are included at the end of the book.

1.1 Motivation

Let us consider a stochastic differential equation on $\mathbb{R}^n$ of the type

$$X_t^{x_0} = x_0 + \sum_{i=1}^{d} \int_0^t V_i(X_s^{x_0}) \circ dB_s^i, \; t \geq 0, \tag{1.1}$$

where:

(1) $x_0 \in \mathbb{R}^n$;
(2) $V_1, ..., V_d$ are C^∞ bounded vector fields on $\mathbb{R}^n$;
(3) $\circ$ denotes Stratonovitch integration;
(4) $(B_t)_{t\geq 0} = (B_t^1, ..., B_t^d)_{t\geq 0}$ is a d-dimensional standard Brownian motion.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function and denote by $(X_t^{x_0})_{t\geq 0}$ the solution of (2.1) with initial condition $x_0 \in \mathbb{R}^n$. First, by Itô's formula, we have

$$f(X_t^{x_0}) = f(x_0) + \sum_{i=1}^{d} \int_0^t (V_i f)(X_s^{x_0}) \circ dB_s^i, \ t \geq 0.$$

Now, a new application of Itô's formula to $V_i f(X_s^x)$ leads to

$$f(X_t^{x_0}) = f(x_0) + \sum_{i=1}^{d} (V_i f)(x_0) B_t^i + \sum_{i,j=1}^{d} \int_0^t \int_0^s (V_j V_i f)(X_u^{x_0}) \circ dB_u^j \circ dB_s^i.$$

We can continue this procedure to get after N steps

$$f(X_t^{x_0}) = f(x_0) + \sum_{k=1}^{N} \sum_{I=(i_1,\ldots i_k)} (V_{i_1} \ldots V_{i_k} f)(x_0) \int_{\Delta^k[0,t]} \circ dB^I + \mathbf{R}_N(t),$$

for some remainder term $\mathbf{R}_N$, where we used the notations:

(1)

$$\Delta^k[0,t] = \{(t_1, \ldots, t_k) \in [0,t]^k, t_1 \leq \ldots \leq t_k\};$$

(2) If $I = (i_1, \ldots i_k) \in \{1, \ldots, d\}^k$ is a word with length k,

$$\int_{\Delta^k[0,t]} \circ dB^I = \int_{0 \leq t_1 \leq \ldots \leq t_k \leq t} \circ dB_{t_1}^{i_1} \circ \ldots \circ dB_{t_k}^{i_k}.$$

If we dangerously do not care about convergence questions (these questions are widely discussed in [Ben Arous (1989b)]), it is tempting to let $N \to +\infty$ and to assume that $\mathbf{R}_N \to 0$. We are thus led to the nice (but formal!) formula

$$f(X_t^{x_0}) = f(x_0) + \sum_{k=1}^{+\infty} \sum_{I=(i_1,\ldots i_k)} (V_{i_1} \ldots V_{i_k} f)(x_0) \int_{\Delta^k[0,t]} \circ dB^I. \tag{1.2}$$

We can rewrite this formula in a more convenient way. Let Φ_t be the stochastic flow associated with the stochastic differential equation (2.1). There is a natural action of Φ_t on smooth functions: The pull-back action given by

$$(\Phi_t^* f)(x_0) = (f \circ \Phi_t)(x_0) = f(X_t^{x_0}).$$

The formula (1.2) shows then that we have the following formal development for this action

$$\Phi_t^* = \mathbf{Id} + \sum_{k=1}^{+\infty} \sum_{I=(i_1,\dots i_k)} V_{i_1}\dots V_{i_k} \int_{\Delta^k[0,t]} \circ dB^I. \tag{1.3}$$

Though this formula does not make sense from an analytical point of view, at least, it shows that the *probabilistic information* contained in the stochastic flow associated with the stochastic differential equation (1.1) is given by the set of Stratonovitch chaos $\int_{\Delta^k[0,t]} \circ dB^I$. What is a priori less clear is that the *algebraic information* which is relevant for the study of Φ_t^* is given by the structure of the Lie algebra generated by the $V_i's$, and this is precisely this aspect we want to stress in this chapter which is devoted to the study of formal objects like

$$\mathbf{Id} + \sum_{k=1}^{+\infty} \sum_{I=(i_1,\dots i_k)} V_{i_1}\dots V_{i_k} \int_{\Delta^k[0,t]} \circ dB^I.$$

Such objects and their relations with flows seem to appear the first time in the works of K.T. Chen [Chen (1957)], [Chen (1961)].

1.2 The signature of a Brownian motion

Let us denote by $\mathbb{R}[[X_1, \dots, X_d]]$ the **non-commutative** algebra of formal series with d indeterminates.

Definition 1.1 The signature of a d-dimensional standard Brownian motion $(B_t)_{t\geq 0}$ is the element of $\mathbb{R}[[X_1, \dots, X_d]]$ defined by

$$S(B)_t = \mathbf{1} + \sum_{k=1}^{+\infty} \sum_{I=(i_1,\dots i_k)} X_{i_1}\dots X_{i_k} \int_{\Delta^k[0,t]} \circ dB^I, \ t \geq 0.$$

Remark 1.1 *We define the signature by using Stratonovitch's integrals because we keep in mind the connection with stochastic flows which appeared with formula (1.3). Nevertheless, it is possible to define a signature by using Itô's integrals. The link between these two signatures is given in Proposition 1.2 below.*

Remark 1.2 *In the same way, it is of course also possible to define the signature of a general semimartingale.*

Observe that the signature hence defined is the solution of the *formal* stochastic differential equation

$$S(B)_t = \mathbf{1} + \sum_{i=1}^{d} \int_0^t S(B)_s X_i \circ dB_s^i, \ t \geq 0. \tag{1.4}$$

Such linear equations appear in the study of Brownian motions on Lie groups. Indeed, let $\mathbb{G}$ be a Lie group with Lie algebra $\mathfrak{g}$.

Definition 1.2 A process $(X_t)_{t\geq 0}$ with values in $\mathbb{G}$ is called a (left) Brownian motion on $\mathbb{G}$ if:

(1) $(X_t)_{t\geq 0}$ is continuous;
(2) for each $s \geq 0$, the process $\left(X_s^{-1} X_{t+s}\right)_{t\geq 0}$ is independent of the process $(X_u)_{0\leq u\leq s}$;
(3) for each $s \geq 0$, the processes $\left(X_s^{-1} X_{t+s}\right)_{t\geq 0}$ and $(X_t)_{t\geq 0}$ are identical in law.

In a general way, one can construct Brownian motions on Lie groups by solving differential equations. Let us consider $V_1, ..., V_d \in \mathfrak{g}$. As explained in Appendix B, $V_1, ..., V_d \in \mathfrak{g}$ can be seen as left invariant vector fields on $\mathbb{G}$, so that we can consider the following stochastic differential equation

$$X_t = 1_{\mathbb{G}} + \sum_{i=1} \int_0^t V_i(X_s) \circ dB_s^i, \ t \geq 0, \tag{1.5}$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion on $\mathbb{R}^d$. For instance, if $\mathbb{G}$ is a linear group of matrices, equation (1.5) can be rewritten

$$X_t = 1_{\mathbb{G}} + \sum_{i=1} \int_0^t X_s V_i \circ dB_s^i.$$

It is easily seen that there is a unique solution $(X_t)_{t\geq 0}$ to the stochastic differential equation (1.5), and this solution is a (left) Brownian motion on $\mathbb{G}$. The process $(X_t)_{t\geq 0}$ is called a lift of $(B_t)_{t\geq 0}$ in $\mathbb{G}$. It is interesting to note that, conversely, each Brownian motion on $\mathbb{G}$ is solution of a stochastic differential equation

$$X_t = X_0 + \int_0^t V_0(X_s) ds + \sum_{i=1} \int_0^t V_i(X_s) \circ dB_s^i, \ t \geq 0,$$

where $V_0, V_1, ..., V_d$ are left-invariant vector fields on $\mathbb{G}$; for further details on this, we refer to [Hunt (1958)] and [Yosida (1952)].