

# Contents

<b>Preface</b>	<b>xiii</b>
<b>1 Galileo</b>	<b>1</b>
1.1 Principle of Galilean relativity . . . . .	1
1.2 Galilean transformations . . . . .	2
1.3 Lie group actions of $SE(3)$ & $G(3)$ . . . . .	8
<b>2 Newton, Lagrange, Hamilton</b>	<b>13</b>
2.1 Newton . . . . .	13
2.1.1 Newtonian form of free rigid rotation . . . . .	13
2.1.2 Newtonian form of rigid-body motion . . . . .	23
2.2 Lagrange . . . . .	28
2.2.1 The principle of stationary action . . . . .	28
2.3 Noether's theorem . . . . .	30
2.3.1 Lie symmetries & conservation laws . . . . .	30
2.3.2 Infinitesimal transformations of a Lie group . . . . .	31
2.4 Lagrangian form of rigid-body motion . . . . .	39
2.4.1 Hamilton-Pontryagin variations . . . . .	43
2.4.2 Manakov's formulation of the $SO(n)$ rigid body . . . . .	45
2.4.3 Matrix Euler-Poincaré equations . . . . .	46
2.4.4 Manakov's integration of the $SO(n)$ rigid body . . . . .	47
2.5 Hamilton . . . . .	49
2.5.1 Hamiltonian form of rigid-body motion . . . . .	50
2.5.2 Lie-Poisson Hamiltonian rigid-body dynamics . . . . .	51
2.5.3 Nambu's $\mathbb{R}^3$ Poisson bracket . . . . .	53
2.5.4 Clebsch variational principle for the rigid body . . . . .	56

<b>3</b>	<b>Quaternions</b>	<b>61</b>
3.1	Operating with quaternions . . . . .	62
3.1.1	Quaternion multiplication using Pauli matrices	62
3.1.2	Quaternionic conjugate . . . . .	64
3.1.3	Decomposition of three-vectors . . . . .	67
3.1.4	Alignment dynamics for Newton's 2nd Law . .	68
3.1.5	Quaternionic dynamics of Kepler's problem . .	71
3.2	Quaternionic conjugation . . . . .	74
3.2.1	Quaternionic conjugation in CK terms . . . . .	74
3.2.2	Pure quaternions, Pauli matrices & $SU(2)$ . . .	79
3.2.3	Tilde map: $\mathbb{R}^3 \simeq su(2) \simeq so(3)$ . . . . .	81
3.2.4	Pauli matrices and Poincaré's sphere $\mathbb{C}^2 \rightarrow S^2$	82
3.2.5	Poincaré's sphere and Hopf's fibration . . . . .	83
<b>4</b>	<b>Quaternionic conjugacy</b>	<b>87</b>
4.1	Cayley-Klein dynamics . . . . .	87
4.1.1	Cayley-Klein parameters, rigid-body dynamics	87
4.1.2	Body angular frequency . . . . .	89
4.1.3	Hamilton's principle in CK parameters . . . . .	90
4.2	Actions of quaternions . . . . .	91
4.2.1	AD, Ad, ad, Ad* & ad* actions of quaternions	92
4.2.2	AD-, Ad- & ad- for Lie algebras and groups . .	92
<b>5</b>	<b>Special orthogonal group</b>	<b>101</b>
5.1	Adjoint and coadjoint actions of $SO(3)$ . . . . .	101
5.1.1	Ad and ad operations for the hat map . . . . .	101
5.1.2	AD, Ad & ad actions of $SO(3)$ . . . . .	103
5.1.3	Dual Lie algebra isomorphism $\sim : so(3)^* \rightarrow \mathbb{R}^3$ .	104
<b>6</b>	<b>The special Euclidean group</b>	<b>109</b>
6.1	Introduction to $SE(3)$ . . . . .	109
6.2	Adjoint operations for $SE(3)$ . . . . .	111
6.2.1	AD operation for $SE(3)$ . . . . .	111
6.2.2	Ad-operation for $SE(3)$ . . . . .	111
6.2.3	Ad*-operation for $SE(3)$ . . . . .	112
6.3	Adjoint actions of $se(3)$ . . . . .	114
6.3.1	The ad action of $se(3)$ on itself . . . . .	114

6.3.2	The $\text{ad}^*$ action of $\mathfrak{se}(3)$ on its dual $\mathfrak{se}(3)^*$ . . .	116
6.4	Left versus Right . . . . .	118
6.4.1	Left-invariant tangent vectors . . . . .	118
6.4.2	The special Euclidean group $SE(2)$ . . . . .	120
<b>7</b>	<b>Geometric Mechanics on <math>SE(3)</math></b>	<b>123</b>
7.1	Left-invariant Lagrangians . . . . .	123
7.1.1	Legendre transform from $\mathfrak{se}(3)$ to $\mathfrak{se}(3)^*$ . . . .	125
7.1.2	Lie-Poisson bracket on $\mathfrak{se}(3)^*$ . . . . .	126
7.1.3	Coadjoint motion on $\mathfrak{se}(3)^*$ . . . . .	127
7.2	Kirchhoff equations . . . . .	129
7.2.1	Looks can be deceiving: The heavy top . . . .	131
<b>8</b>	<b>Heavy top equations</b>	<b>133</b>
8.1	Introduction and definitions . . . . .	133
8.2	Heavy top action principle . . . . .	134
8.3	Lie-Poisson brackets . . . . .	136
8.3.1	Lie-Poisson brackets and momentum maps . .	136
8.3.2	The heavy top Lie-Poisson brackets . . . . .	137
8.4	Clebsch action principle . . . . .	139
8.5	Kaluza-Klein construction . . . . .	140
<b>9</b>	<b>The Euler-Poincaré theorem</b>	<b>143</b>
9.1	Action principles on Lie algebras . . . . .	143
9.2	Hamilton-Pontryagin principle . . . . .	146
9.3	Clebsch approach to Euler-Poincaré . . . . .	147
9.3.1	Defining the Lie derivative . . . . .	149
9.3.2	Clebsch Euler-Poincaré principle . . . . .	150
9.4	Lie-Poisson Hamiltonian formulation . . . . .	152
9.4.1	Cotangent-lift momentum maps . . . . .	153
<b>10</b>	<b>Lie-Poisson Hamiltonian form</b>	<b>155</b>
10.1	Hamiltonian continuum spin chain . . . . .	156
<b>11</b>	<b>Momentum maps</b>	<b>165</b>
11.1	The standard momentum map . . . . .	165
11.2	Cotangent lift . . . . .	167
11.3	Examples . . . . .	169

<b>12 Round rolling rigid bodies</b>	<b>181</b>
12.1 Introduction . . . . .	182
12.1.1 Holonomic versus nonholonomic . . . . .	182
12.1.2 Chaplygin's top . . . . .	183
12.2 Hamilton-Pontryagin principle . . . . .	187
12.2.1 HP principle for Chaplygin's top . . . . .	190
12.2.2 Circular disk rocking in a vertical plane . . . . .	194
12.2.3 Euler's rolling and spinning disk . . . . .	197
12.3 Nonholonomic symmetry reduction . . . . .	203
12.3.1 Semidirect-product structure . . . . .	203
12.3.2 Euler-Poincaré theorem . . . . .	206
<b>A Geometrical structure</b>	<b>213</b>
A.1 Manifolds . . . . .	213
A.2 Motion: Tangent vectors and flows . . . . .	221
A.2.1 Vector fields, integral curves and flows . . . . .	222
A.2.2 Differentials of functions: The cotangent bundle	224
A.3 Tangent and cotangent lifts . . . . .	225
A.3.1 Summary of derivatives on manifolds . . . . .	226
<b>B Lie groups and Lie algebras</b>	<b>229</b>
B.1 Matrix Lie groups . . . . .	229
B.2 Defining matrix Lie algebras . . . . .	234
B.3 Examples of matrix Lie groups . . . . .	235
B.4 Lie group actions . . . . .	237
B.4.1 Left and right translations on a Lie group . . . . .	239
B.5 Tangent and cotangent lift actions . . . . .	240
B.6 Jacobi-Lie bracket . . . . .	243
B.7 Lie derivative and Jacobi-Lie bracket . . . . .	245
B.7.1 Lie derivative of a vector field . . . . .	245
B.7.2 Vector fields in ideal fluid dynamics . . . . .	247
<b>C Enhanced coursework</b>	<b>249</b>
C.1 Variations on rigid-body dynamics . . . . .	249
C.1.1 Two times . . . . .	249
C.1.2 Rotations in complex space . . . . .	254
C.1.3 Rotations in four dimensions: $SO(4)$ . . . . .	258

<i>CONTENTS</i>	xi
C.2 $\mathbb{C}^3$ oscillators . . . . .	262
C.3 $GL(n, \mathbb{R})$ symmetry . . . . .	267
<b>Bibliography</b>	<b>273</b>
<b>Index</b>	<b>289</b>