Chapter 1

Introduction

This book contains an account of the notions involved in constructing a theory of supermanifolds and the associated machinery and techniques of differential geometry, together with applications to various areas of physics, including supersymmetry and the quantization of systems with symmetry, and to classical geometry.

The concept of supermanifold involves an extension of a classical manifold to include some notion of anticommuting coordinate; indeed more generally the prefix 'super' is used with many mathematical objects to denote an extension from commutativity to graded commutativity, or to a controlled mixture of both commutativity and anticommutativity. The study of supermanifolds involves mathematical ideas from geometry, analysis, algebra and topology. While much of the original motivation came from particle physics, the concepts and language of supermanifolds have proved powerful in many parts of theoretical physics and pure mathematics, and the range of influence continues to broaden.

Historically anticommuting variables, and some of the constructions now distinguished by the prefix 'super', appeared in mathematics many years before the development of supersymmetry in physics triggered an explosion of interest in super mathematics. Of course anticommuting objects appear in many areas of geometry and algebra, examples include differential forms, the use of the exterior algebra over a Lie algebra in Lie algebra cohomology and the Weil model of equivariant cohomology. But perhaps the earliest step in 'super' mathematics was Cartan's recognition that a Clifford algebra could be represented on a Grassmann algebra if one included a notion of differentiation with respect to a generator as well as multiplication [29], an idea that was to reappear decades later in connection with fermion anticommutation relations. In his seminal work on quantum fields Schwinger

[139] introduced anticommuting variables in order to extend to fermions his treatment of quantum fields using Green's functions and sources. Differential calculus for functions of anticommuting variables was introduced by Martin [102] who extended Feynman's path-integral method of quantization to systems containing fermions and thus needed a 'classical' fermion to quantize. Anticommuting variables were used by a number of other authors to develop fermionic quantization in close analogy to methods for bosonic quantization using conventional, commuting variables; an extensive and pioneering study was made by Berezin [16].

A supersymmetric theory enjoys invariance under a symmetry which exchanges bosonic and fermionic degrees of freedom; as a result, an approach which treats fermions and bosons on an equal footing (as is the case when commuting and anticommuting variables are used) is likely to be particularly useful, and it has indeed been the case that super mathematical ideas have proved effective in the study of supersymmetry. Where geometrical ideas are involved in a supersymmetric model the anticommuting extension must respect and possibly develop this geometry, and thus what have become known as supermanifolds are required, together with much of the machinery of differential geometry. As interest in supersymmetric models took off in the physics community following the appearance of the pioneering models in the early 1970's [152, 158], there was a correspondingly rapid development of super geometry and other areas of super mathematics. The importance of anticommuting variables in supersymmetry can also be seen quite independently of any specific treatment of fermions, by considering the nature of the group of symmetries involved; at the infinitesimal level these form a super Lie algebra, that is, an algebra whose generators can be classified as either odd or even, and which closes under commutation of even generators and anticommutation of odd generators; the natural way to regard a group made from such generators is to associate commuting and anticommuting parameters respectively with the even and odd generators, leading to the concept of super Lie group. Although it is possible to handle supersymmetry without using anticommuting variables, their use often suggests by analogy some new approach to be tried, and has been a fruitful source of both conceptual and technical ideas.

At its simplest super mathematics extends classical ideas to a Z_2 graded setting, introducing a notion of even and odd, together with a rule that an extra sign factor appears whenever two odd elements are interchanged. Some of the development proceeds by a straightforward analogy with the classical, purely commuting case, with little more required than the insertion of the correct sign factors; however one aim of this book is to make it clear that the interesting and powerful parts of super mathematics are those where a straightforward analogy with classical mathematics is not possible, or does not give a full picture. There are various characteristic features of super mathematics which particularly stand out, which will occur repeatedly in the course of the book. These include the notion of super derivative (which provides a square root of a conventional derivative and also allows a representation of canonical anticommutation relations and Clifford algebras) and the Berezin integral which preserves certain characteristics of classical integration, but also has unexpected but valuable features. There is also the concept of supertrace, which leads to cancellations between odd and even sectors. These features, which are interrelated, are the key ingredients of many application of super mathematics both in geometry and in theoretical physics.

In super geometry there are two rather different, but essentially equivalent, approaches to supermanifold. In the first, which will be referred to as the *concrete* approach, a supermanifold is a set, more specifically it is a manifold modelled on some flat 'superspace' so that it has local coordinates some of which take values in the even and some in the odd part of a Grassmann algebra. In the second approach to supermanifolds, which will be referred to as the *algebro-geometric* approach, it is the sheaf of functions on a manifold which is extended, rather than the manifold itself. Here super geometry is distinguished from more general non-commutative geometry in which only an algebro-geometric approach seems to exist, with the non-commutativity expressed in terms of rings of 'functions'.

In this book both approaches to supermanifold are described. Large swathes of the subject are independent of the approach, but the emphasis in this book is on the concrete approach, because of the nature of the applications considered. Each approach has its protagonists, but in general a multicultural point of view, using the language of whichever of the two approaches best suits the matter in hand, is possible because there is a precise correspondence between algebro-geometric supermanifolds and concrete supermanifolds, as is explained in Chapter 8. Given this choice of approach, which does not exist for more general non-commutative geometry, it seems sensible to exploit all possibilities. The algebro-geometric approach has greater mathematical elegance and simplicity, because in its simplest form there is no need to introduce an auxiliary Grassmann algebra. However certain useful concepts, such as a point in a supermanifold, or an odd constant, are more complicated to define. To an accomplished algebraic geometer this will not not be a problem, but to many who might use supermanifolds this adds an unnecessary complication when a more direct approach is possible. Moreover, in may physical applications it is in fact necessary to introduce an auxiliary Grassmann algebra (or extra odd dimensions) when using the algebro-geometric approach, and so the purity of the approach is diluted. The concrete approach, in which a supermanifold is a set, and a super Lie group is a group, has a psychological advantage in some contexts in suggesting analogies with steps taken in classical differential geometry. It also allows rather simpler terminology, for instance when using functions between supermanifolds, and so makes it easier to give a direct description of various applications and techniques. But it must be emphasised that this is a question of choice of language, not an intrinsic difference. As an example, consider the $(1,1)$-dimensional super group of super translations. Anticipating some notation and terminology, this is readily described in the concrete approach as the set $\mathbb{R}_S^{1,1}$ with group action

$$(x;\xi) \circ (y;\eta) = (x + y + \xi\eta; \xi + \eta)\,,$$

while in the algebro-geometric approach the same object is captured in a less direct way. The main objection to the concrete approach is that it carries the extra baggage of a Grassmann algebra whose individual elements do not individually signify as much as they might appear to; the myriad coefficients, real or complex, of the Grassmann algebra expanded with respect to some basis, do not carry useful information. While this is true, broadly speaking the notion of concrete supermanifold is independent of the choice of Grassmann algebra since a particular topology (due to DeWitt [43]), which does not distinguish between the various nilpotent elements of the Grassmann algebra, is used. One distils out the meaninglessness of the Grassmann detail by showing (following Batchelor [12]) that there is a natural sequence of supermanifolds modelled on Grassmann algebras with increasing numbers of generators, and taking the inductive limit.

As remarked above, there are certain areas of supermanifold theory where an auxiliary Grassmann algebra (or equivalent) is required in both approaches. While in the smooth setting a theorem due to Batchelor [11] shows that the data of a supermanifold is simply that of a vector bundle over a classical manifold, in the complex setting one immediately encounters supermanifolds whose data includes anticommuting parameters or moduli. To handle such supermanifolds in the algebro-geometric approach requires either the introduction of an auxiliary Grassmann algebra or the consideration of families rather than individual supermanifolds, in which case auxiliary coordinates (even, odd or both) are required. In physical applications, and also in applications to classical geometry, the principal mechanism for extracting a real or complex number from the nilpotent superfluity is the Berezin integral. This a formal algebraic integral, not the limit of a sum depending on the details of any Grassmann algebra element. It is closely related to the notion of supertrace, as is explained in Section 11.1.

The use of supergeometric language, concepts and constructions is in a sense optional; the effect of adding odd dimensions and so on can always, or at least almost always, be produced by an alterative mathematical construction, and one does not really expect that anticommuting variables will model reality in the same way that real variables seem to. A strong argument for taking the 'super' point of view is that it suggests new approaches by analogy with procedures in classical mathematics. In a number of important situations super methods have proved very powerful. These include applications to classical geometry, supersymmetry and and the treatment of systems with gauge symmetry using ghosts and BRST cohomology. In this context it is interesting to quote Voronov's comment [153] in connection with the Faddeev-Popov quantization of gauge fields [49] that 'although he obtained similar results, DeWitt was unable to present them in a convenient form because he was unaware of the concept of the Berezin integral; see the author's forward to the Russian translation [42]' of [41].

Although a number of books and articles on supermanifolds have appeared, most are devoted to one particular approach, with the majority taking the algebro-geometric approach. In this book, while mostly using the language of the concrete approach, the aim is to take stock of a wide range of ideas and applications in supermanifold theory, hopefully throwing light on a concept that over the years has proved robust and adaptable to a number of purposes and abstracting the essential ideas from the various slightly different approaches. In general the style adopted has not assumed great mathematical sophistication on the part of the reader; this may irritate the accomplished mathematician, but is intended to increase the accessibility of the material for those who might use it in a variety of contexts. Although this book aims to provide a reasonably comprehensive survey of supermanifolds and their uses, examples considered are selected for their interest or their usefulness and there is no attempt to provide a complete taxonomy.

While detailed references will of course be given as the book proceeds, it seems appropriate to mention in the introduction some of the major landmarks in the history of the subject, as well as the principal books. Supermanifolds were introduced into the mathematical literature by Berezin and Leĭtes [19] and by Kostant [95], who used the term graded manifold. Both these authors used the algebro-geometric approach. Inspired by physics, DeWitt developed the theory of supermanifolds in the concrete approach, introducing the crucial topology which underpins the relation between the algebro-geometric and concrete approach [43]; Rogers developed the concrete approach from a slightly different point of view, imitating as closely as possible the construction of a classical manifold [118]. Berezin's extensive work on super mathematics was collected together after his untimely death in 1980 [17]. An early review article is that of Leĭtes [99]. More recent books on supermanifolds include that of Bartocci, Bruzzo and Hernández Ruipérez [8] and that of Tuynman [149]. Supermanifolds in the algebro-geometric approach are a central topic in the book of Manin [100]. A treatment of supermanifolds may be found in a section by Deligne and Morgan (following Bernstein) in 'Quantum fields and strings: a course for mathematicians' [39]. Two key papers by Batchelor [11, 12] provide the key structure theorem for smooth supermanifolds and establish the relationship between the concrete and the algebro-geometric approach.

Many of the basic features of supermanifold theory are well understood, although there are areas, such as a systematic treatment of subsupermanifolds, which remain incomplete. The structures developed have proved useful in may contexts; one result of supermanifold theory is simply to confirm that a heuristic approach, using super ideas developed by simple analogy with classical ones, is generally valid and often, when combined with the features of super geometry which have no classical analogue, very powerful. In many cases it is not the fully general theory of supermanifolds which has proved of interest, rather it has been supermanifolds restricted by some particular condition, for instance the super covariant condition used to define a super Riemann surface or the dimensional restriction necessary for an odd symplectic supermanifold, which have proved fruitful. It seems likely that further developments of supermanifold theory will involve new structures of this nature.