Chapter 1

MATHEMATICAL PRELUDE

In this chapter we have collected some of the standard results from the fields of Algebra, Functional Analysis, and Representation Theory which are well known to the specialist working in those fields, but which may not be necessarily assumed to be common knowledge for physicists.

1.1 Algebra

Let $\mathcal{A} \neq \{0\}$ be a vector space over the complex numbers $\mathcal{C}$. $\mathcal{A}$ is called an associative algebra with unity over $\mathcal{C}$, if a product $\mathcal{A} \times \mathcal{A} \to \mathcal{A}, (A, B) \mapsto AB$ is defined on $\mathcal{A}$ such that

$$\begin{aligned} (AB)C &= A(BC), \\ A(B+C) &= AB + AC, \\ (A+B)C &= AC + BC, \\ (\alpha A)B &= A(\alpha B) = \alpha AB, \text{ for } \alpha \in \mathcal{C}, \end{aligned}$$

and if there exists an element $I \in \mathcal{A}$ such that $IA = AI = A$ for all $A \in \mathcal{A}$.

A set $\mathcal{G}$ of elements of $\mathcal{A}$ is called a system of generators of $\mathcal{A}$ if the smallest closed subalgebra with unity containing $\mathcal{G}$ coincides with $\mathcal{A}$. The unity I is not included in the system of generators.

Let us assume that the associative algebra with unity $\mathcal{A}$ is generated by d elements $X_1, \ldots, X_d$, i.e. $\mathcal{G} = \{X_i\}_{i=1}^d$. Then each element of $\mathcal{A}$ can be written as

$$\begin{aligned} A &= a^0 I + \sum_{i=1}^{d} a^i X_i + \sum_{i,j=1}^{d} a^{ij} X_i X_j + \ldots \\ & a^0, a^i, a^{ij}, \ldots \in \mathcal{C}. \end{aligned} \tag{1.1}$$

We restrict the discussion to the case of a finite number of generators and we will not discuss topologies of $\mathcal{A}$, i.e. it is assumed that the above sums for every A are arbitrarily large but finite. **Defining algebraic relations** are relations among the generators

$$P(X_1, \ldots, X_d) = 0, \tag{1.2}$$

where $P(x_1, \ldots, x_d)$ is a polynomial of d variables with complex coefficients. Let $B \in \mathcal{A}$ be represented by

$$\begin{aligned} B &= b^0 I + \sum_{i=1}^{d} b^i X_i + \sum_{i,j=1}^{d} b^{ij} X_i X_j + \ldots \\ & b^0, b^i, b^{ij}, \ldots \in \mathcal{C}. \end{aligned} \tag{1.3}$$

If one can bring (1.3) into the same form as (1.1) with the same coefficients $a^0, a^i, \ldots$ by using the defining algebraic relations (1.2), then B is equal to A.