1. Quantum Mechanics

Macroscopic quantum phenomena are exceedingly attractive and important. A minimum review of quantum mechanics is given from this point of view. The creation and annihilation operators of particles or excitation modes are introduced. Quantum field theory is constructed on the basis of these operators. An emphasis is placed on the minimum-uncertainty state, that is a coherent state, where the particle number and the conjugate phase are simultaneously measurable most accurately as a quantum system.

1.1 Hilbert Spaces

A state is represented by an element $|\mathbb{S}\rangle$ in a Hilbert space $\mathbb{H}$. A Hilbert space $\mathbb{H}$ is a vector space characterized by the following properties:

1. *Principle of superposition*
   If $|\mathbb{S}_1\rangle$ and $|\mathbb{S}_2\rangle$ are elements of the space $\mathbb{H}$ then so is $\lambda_1|\mathbb{S}_1\rangle + \lambda_2|\mathbb{S}_2\rangle$ for arbitrary complex numbers λ_1 and λ_2.
2. *Inner product*
   A complex number $\langle\mathbb{S}_2|\mathbb{S}_1\rangle$, called *inner product*, is assigned to any pair of states. It obeys

$$\langle\mathbb{S}_2|\mathbb{S}_1\rangle = \langle\mathbb{S}_1|\mathbb{S}_2\rangle^*, \tag{1.1.1a}$$

$$\langle\mathbb{S}_3|(\lambda_1|\mathbb{S}_1\rangle + \lambda_2|\mathbb{S}_2\rangle) = \lambda_1\langle\mathbb{S}_3|\mathbb{S}_1\rangle + \lambda_2\langle\mathbb{S}_3|\mathbb{S}_2\rangle, \tag{1.1.1b}$$

$$\langle\mathbb{S}|\mathbb{S}\rangle \geq 0. \tag{1.1.1c}$$

   The inner product $\langle\mathbb{S}|\mathbb{S}\rangle$ is called the norm of the state $|\mathbb{S}\rangle$. It vanishes if and only if $|\mathbb{S}\rangle = 0$.

When an operator $\mathcal{O}$ is given, a complex number $\langle\mathbb{S}_2|\mathcal{O}|\mathbb{S}_1\rangle$ is determined for any pair of states $|\mathbb{S}_1\rangle$ and $|\mathbb{S}_2\rangle$ as the inner product of two states $|\mathbb{S}_2\rangle$ and $\mathcal{O}|\mathbb{S}_1\rangle$. Conversely, the operator $\mathcal{O}$ is uniquely defined when a complex number $\langle\mathbb{S}_2|\mathcal{O}|\mathbb{S}_1\rangle$ is given to any pair of states. Thus, a new operator $\mathcal{O}^\dagger$ is defined from $\mathcal{O}$ by requiring

$$\langle \mathbb{S}_2|\mathcal{O}^\dagger|\mathbb{S}_1\rangle = \langle \mathbb{S}_1|\mathcal{O}|\mathbb{S}_2\rangle^* \tag{1.1.2}$$

for any pair of states, where $\langle \mathbb{S}_1|\mathcal{O}|\mathbb{S}_2\rangle^*$ is the complex conjugate of $\langle \mathbb{S}_1|\mathcal{O}|\mathbb{S}_2\rangle$. The operator $\mathcal{O}^\dagger$ is called the *Hermitian conjugate* of $\mathcal{O}$.

A measurement of the operator $\mathcal{O}$ made on the state $|\mathbb{S}\rangle$ yields an expectation value $\langle \mathbb{S}|\mathcal{O}|\mathbb{S}\rangle$. Since we get a real number from the measurement of a physical quantity, the physical operator should satisfy $\mathcal{O}^\dagger = \mathcal{O}$: such an operator is called a *Hermitian operator*. On the other hand, when an operator $\mathcal{O}$ satisfies $\mathcal{O}^\dagger = \mathcal{O}^{-1}$, where $\mathcal{O}^{-1}$ is the inverse of $\mathcal{O}$, it is called a *unitary operator*.

For a Hermitian operator H, the solutions of the eigenvalue problem,

$$H|\mathbb{S}_i\rangle = \varepsilon_i|\mathbb{S}_i\rangle, \tag{1.1.3}$$

may be chosen to satisfy the orthonormality condition

$$\langle \mathbb{S}_j|\mathbb{S}_k\rangle = \delta_{jk}, \tag{1.1.4}$$

and the completeness condition

$$\sum_j |\mathbb{S}_j\rangle\langle \mathbb{S}_j| = 1. \tag{1.1.5}$$

The states $|\mathbb{S}_j\rangle$ form a basis of a Hilbert space $\mathbb{H}$.

1.2 Hamiltonian Formalism

The dynamics of a system is determined when the Lagrangian L is given. It is a function of a collection of "coordinates", $q \equiv \{q_1, q_2, \cdots, q_N\}$, with their "velocities" $\dot{q}$ at time t. The action is defined by

$$S = \int_{t_1}^{t_2} dt\, L[q(t), \dot{q}(t)]. \tag{1.2.1}$$

The equations of motion follow from a principle of least action: They are the Euler-Lagrange equation,

$$\frac{\delta S}{\delta q(t)} \equiv \frac{\partial L}{\partial q(t)} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}(t)} = 0. \tag{1.2.2}$$

The conjugate "momentum" (canonical momentum) is defined by

$$p_i = \frac{\partial L(q, \dot{q})}{\partial \dot{q}_i}. \tag{1.2.3}$$