Chapter 1

INTRODUCTION

This chapter covers informally material that will be treated extensively in the book. Some of it may have already been learned in undergraduate mechanics. Introduction or review as they may be, the following pages are meant to lead into the substance of analytical mechanics.

1.1 Motion in phase space

It is known from General Physics that the dynamics of particles and rigid bodies is governed by second order differential equations. For computational purposes, it is convenient to replace these by systems of first order equations. Thus Newton's second law in the form

$$m\ddot{x} = f(x, \dot{x}, t) \tag{1.1}$$

is replaced by the system

$$\dot{p} = f(q, p, t) \quad , \quad \dot{q} = p \quad , \tag{1.2}$$

where we have renamed the coordinate ($q = x$), denoted by p the momentum, and taken the mass equal to unity.

This technical artifice is epitomized by Hamilton's equations for the generalized coordinates $q_1, \ldots q_n$ and their conjugate momenta $p_1, \ldots p_n$,

$$\dot{p}_k = -\partial H(p, q, t)/\partial q_k \quad , \tag{1.3}$$

$$\dot{q}_k = \partial H(p, q, t)/\partial p_k \quad , \tag{1.4}$$

where H is the "Hamiltonian" (see chapter 7).

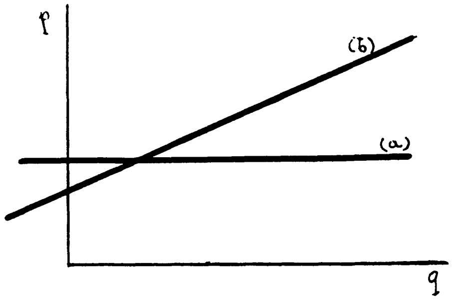

Figure 1.1: Arbitrariness of p

The above process is not unique. For example, for a free particle ($\ddot{q} = 0$) rather than

$$\dot{p} = 0 \quad , \quad \dot{q} = p \quad , \tag{1.5}$$

according to which p is constant and $q = pt + q_0$ (figure 1.1(a)), we might equivalently have written (figure 1.1(b))

$$\dot{p} = 1 \quad , \quad \dot{q} = p - t \quad , \quad p = t + \alpha \quad , \quad q = \alpha t + \beta \quad . \tag{1.6}$$

As we shall see in Chapter 6, this arbitrariness is connected with one for the Lagrangian.

Once the second order equations have been replaced by systems of first order ones, it is natural to describe the evolution of a system as the motion of a point in a $2n$-dimensional "phase space" for the coordinates $q_1, \ldots q_n$ and the momenta $p_1, \ldots p_n$.

1.2 Motion of a particle in one dimension

It is instructive to study the motion of a particle in one dimension subject to the elastic force $-kq$ and the repulsive force $+kq$ ($k > 0$), respectively.

In the former case, the equations $\dot{p} = -kq$, $\dot{q} = p/m$ have the general solution ($\omega = \sqrt{k/m}$)

$$q = a\cos(\omega t + \alpha) \quad , \quad p = -m\omega a \sin(\omega t + \alpha) \quad , \tag{1.7}$$

so that the representative point in the phase plane (p, q) moves on an ellipse of semi-axes $a = \sqrt{2E/k}$ and $m\omega a = \sqrt{2mE}$ (figure 1.2), where E is the energy. The point $q = 0$, $p = 0$ describes a state of stable equilibrium ($a = 0$, $E = 0$).