



Figure 1.9: Equation (1.34)

## 1.5 The Maupertuis principle

We have seen that

$$S(B) - S(A) = \left( \int_A^B \mathbf{p} \cdot d\mathbf{r} \right)_t = \left( \int_A^B |\mathbf{p}| ds \right)_t, \quad (1.35)$$

where  $\mathbf{p} = \nabla S$ ,  $|\mathbf{p}| = |\nabla S| = \sqrt{2m(E - U(\mathbf{r}))}$  and “t” is the actual trajectory from  $A$  to  $B$ .

Now if “c” is a curve from  $A$  to  $B$  adjacent to the trajectory, we see that

$$S(B) - S(A) = \left( \int_A^B \mathbf{p} \cdot d\mathbf{r} \right)_c \leq \left( \int_A^B \sqrt{2m(E - U(\mathbf{r}))} ds \right)_c, \quad (1.36)$$

since  $\mathbf{p}$  and  $d\mathbf{r}$  are not parallel on “c”.

Hence the Maupertuis principle or principle of “least action”: The action integral from a point  $A$  to a point  $B$ ,  $\int_A^B \sqrt{2m(E - U(\mathbf{r}))} ds$ , is minimal along a trajectory. Less restrictively, Euler’s formulation of the principle states that

$$\delta \int_A^B \sqrt{E - U(\mathbf{r})} ds = 0, \quad (1.37)$$

where  $\delta$  indicates variation of the integral when the coordinates  $\mathbf{r} = (x, y, z)$  of each point of a trajectory from  $A$  to  $B$  are changed to  $\mathbf{r} + \delta\mathbf{r} = (x + \delta x, y + \delta y, z + \delta z)$ , with  $\delta\mathbf{r}$  vanishing at  $A$  and  $B$ .

It must be possible to derive the equations of trajectories from this variational principle. Let us do so in two dimensions and assuming that  $U = U(y)$ . With  $y' \equiv dy/dx$ , we have

$$\delta \int_A^B \sqrt{E - U} ds = \delta \int_A^B \sqrt{E - U(y)} \sqrt{1 + y'^2} dx$$

$$= \int_A^B \left[ -\frac{1}{2} \sqrt{\frac{1+y'^2}{E-U}} \frac{dU}{dy} \delta y + \sqrt{\frac{E-U}{1+y'^2}} y' \frac{d\delta y}{dx} \right] dx = 0 .$$

Partial integration using  $\delta y_A = \delta y_B = 0$  yields the trajectory equation

$$\frac{d}{dx} \left( y' \sqrt{\frac{E-U}{1+y'^2}} \right) + \frac{1}{2} \sqrt{\frac{1+y'^2}{E-U}} \frac{dU}{dy} = 0 .$$

Multiplying by  $y' \sqrt{(E-U)/(1+y'^2)}$  we obtain

$$\frac{d}{dx} \left( \frac{y'^2}{1+y'^2} (E-U) + U \right) = 0 ,$$

so that the quantity in parentheses is a constant, which we denote by  $C$ . Hence

$$y'^2 = \frac{C-U}{E-C} .$$

The value of the constant can be determined from that of  $y'$  at a point of the trajectory. For a projectile ( $U = mgy$ ), we have  $E = mv_0^2/2$ ,  $(y')_{x=y=0} = v_{0y}/v_{0x}$ , and so  $C = mv_{0y}^2/2$ ,  $y' = (\sqrt{v_{0y}^2 - 2gy})/v_{0x}$ . Solution by separation of variables yields equation (1.32) if the value of the integration constant is chosen so that the trajectory may pass through the origin.

## 1.6 The time

What about the time? Equation (1.21) expressed the time interval during which a particle moves from a position to another as the derivative of the action integral with respect to the energy. That formula is easily generalized to three dimensions. In fact

$$\begin{aligned} \frac{\partial}{\partial E} S(\mathbf{r}_2, \mathbf{r}_1, E, \alpha) &= \frac{\partial}{\partial E} \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{p} \cdot d\mathbf{r} = \frac{\partial}{\partial E} \int_{\mathbf{r}_1}^{\mathbf{r}_2} \sqrt{2m(E-U(\mathbf{r}))} ds \\ &= \int_{\mathbf{r}_1}^{\mathbf{r}_2} m ds / \sqrt{2m(E-U(\mathbf{r}))} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} ds/v = t_2 - t_1 , \end{aligned} \quad (1.38)$$

where  $v$  denotes the velocity.

It is interesting to compare this time interval with the time it would take a particle of the same mass, total energy  $E$ , and potential energy  $U(\mathbf{r})$ , to