Integrating over $d\Omega_k$ (while using only 1/8 of the total quadrants) and remembering that for each **k** we have two directions of polarization, we get

$$[1\text{-}30]\quad d\rho = \frac{4\pi}{8}2\frac{V}{\pi^3}k^2dk = \frac{V}{\pi^2}k^2dk = \frac{V}{\pi^2}\frac{8\pi^3}{c^3}\nu^2d\nu = \frac{8\pi V}{c^3}\nu^2d\nu,$$

and the number of modes per unit volume is $(8\pi\nu^2d\nu/c^3)$.

Letting $W(\nu)$ denote the energy density per unit frequency (i.e. energy kT per mode), we get the Rayleigh-Jeans formula:

$$[1\text{-}31]\quad W(\nu) = kT\frac{8\pi\nu^2}{c^3} \quad\Rightarrow\quad W(\nu)d\nu = \frac{8\pi kT}{c^3}\nu^2d\nu.$$

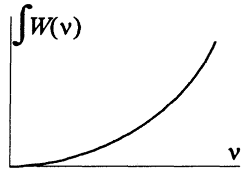

Fig. 1-4

It is immediately clear that we have a problem here. The total energy, obtained by integrating $W(\nu)$, diverges and increases indefinitely (Fig. 1-4), which contradicts experimental results. This, of course, is not surprising since, dimensionally speaking, electromagnetic theory does not provide us with any length measurement, hence we have no knowledge of the λ value for which the curve turns over. If we attempt to use r_0 as the appropriate length, we arrive at a λ in the cosmic ray region, whereas the actual cutoff (turn-over) λ is in the visible region of the spectrum. [One might say that the curve would cut off at some particular λ, referring to the limit within which electromagnetic theory holds. However, it is observed that the cutoff λ is very often in the ultraviolet region, where we know electromagnetic theory to hold.]

Planck's Resolution

Planck did not alter the theory dealing with the number of normal modes per frequency interval, i.e. he did not introduce a length concept. Rather he altered the average energy, $\bar{E}$.

The harmonic oscillator energies are restricted to quantized values, e.g. $E_n = nh\nu$, thereby replacing the continuous distribution by a discrete one. Assuming the distribution to be a Boltzmann distribution,

$$[1\text{-}32]\quad \overline{E} = \frac{\sum_{n=o}^{\infty} E_n e^{-E_n/kT}}{\sum_{n=o}^{\infty} e^{-E_n/kT}} = \frac{h\nu \sum_{n=o}^{\infty} n e^{-nh\nu/kT}}{\sum_{n=o}^{\infty} e^{-nh\nu/kT}}.$$

Fig. 1-5a

Making the substitution $x = \dfrac{h\nu}{kT}$:

$$[1\text{-}33]\quad \overline{E} = \frac{h\nu \sum_{o}^{\infty} n e^{-nx}}{\sum_{o}^{\infty} e^{-nx}} = -h\nu \frac{\partial}{\partial x}\left[\ln \sum_{o}^{\infty} e^{-nx}\right] = -h\nu \frac{\partial}{\partial x}\left[\ln \frac{1}{1 - e^{-x}}\right]$$

$$= h\nu \frac{\partial}{\partial x}\left[\ln(1 - e^{-x})\right] = h\nu\left[\frac{e^{-x}}{1 - e^{-x}}\right] = \frac{h\nu}{e^{x} - 1} = \frac{h\nu}{e^{h\nu/kT} - 1}.$$

Special Cases

(a) $(h\nu/kT) \ll 1$, so that $\sum \rightarrow \int$ (Fig. 1-5a) and $\overline{E} \rightarrow kT$, yielding the Rayleigh-Jeans distribution.

(b) $(h\nu/kT) \gg 1$, $\overline{E} \rightarrow h\nu e^{-h\nu/kT}$ (Fig. 1-5b) so that

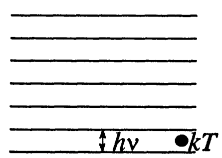

Fig. 1-5b

$$[1\text{-}34]\quad W(\nu)d\nu = \frac{8\pi}{c^3} h\nu^3 e^{-h\nu/kT} d\nu,$$

which is Planck's law and which yields the correct exponential fall-off (Fig. 1-6). In this case, the most probable excited level is 1.

If one makes the correct choice of h, namely, 6.62×10^{-27} erg sec, Planck's law yields a good fit to the experimental data. Hence, one sees that it takes the introduction of h to arrive at a λ value, i.e. the Stefan constant 4.965 is the new number appearing in the theory, arrived at by a proper setting of h.

A basic assumption is that the energy of the field comes in packages of $h\nu$. This was at first taken with a grain of salt.

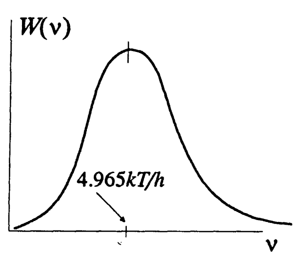

Fig. 1-6