Chapter 1

Introduction

Sophus Lie (1842–1899) and Felix Klein (1849–1925) studied mathematical systems from the perspective of those transformation groups which left the systems invariant. Klein, in his famous "Erlanger" program, pursued the role of finite groups in the studies of regular bodies and the theory of algebraic equations, while Lie developed his notion of continuous transformation groups and their role in the theory of differential equations. Today the theory of continuous groups is a fundamental tool in such diverse areas as analysis, differential geometry, number theory, atomic structure and high-energy physics. In this book we deal with Lie's theorems and extensions thereof, namely its applications to the theory of differential equations.

It is well known that many, if not all, of the fundamental equations of physics are nonlinear and that linearity is achieved as an approximation. One of the important developments in applied mathematics and theoretical physics over the recent years is that many nonlinear equations, and hence many nonlinear phenomena, can be treated as they are, without approximations, and be solved by essentially linear techniques.

One of the standard techniques for solving linear partial differential equations is the Fourier transform. During the past 35 years it was shown that a class of physically interesting nonlinear partial differential equations can be solved by a nonlinear extension of the Fourier technique, namely the inverse scattering transform. This reduces the solution of the Cauchy problem to a series of linear steps. This method, originally applied to the Korteweg-de Vries equation, is now known to be applicable to a large class of nonlinear evolution equations in one space and one time variable, to quite a few equations in $2+1$ dimensions and also to some equations in higher dimensions.

Continuous group theory, Lie algebras and differential geometry play an important role in the understanding of the structure of nonlinear partial differential equations, in particular for generating integrable equations, finding Lax pairs, recursion operators, Bäcklund transformations and finding exact analytic solutions.

Most nonlinear equations are not integrable and cannot be treated via the inverse scattering transform, nor its generalizations. They can of course be treated by numerical methods, which are the most common procedures. Interesting qualitative and quantitative features are however often missed in this manner and it is of great value to be able to obtain, at least, particular exact analytic solutions of nonintegrable equations. Here group theory and Lie algebras play an important role. Indeed, Lie group theory was originally created as a tool for solving ordinary and partial differential equations, be they linear or nonlinear.

New developments have also occurred in this area. Some of them have their origins in computer science. The advent of algebraic computing and the use of such computer languages for symbolic computations such as SymbolicC++, REDUCE, MACSYMA, AXIOM, MAPLE, MATHEMATICA, MuPAD etc., have made it possible (in principle) to write computer programs that construct the Lie algebra of the symmetry group of a differential equation. Other important advances concern the theory of infinite dimensional Lie algebras, such as loop algebras, Kac-Moody and Virasoro algebras which frequently occur as Lie algebras of the symmetry groups of integrable equations in $2+1$ dimensions such as the Kadomtsev-Petviashvili equation. Furthermore, practical and computerizable algorithms have been proposed for finding all subgroups of a given Lie group and for recognizing Lie algebras given their structure constants.

In chapter 2 we give an introduction into group theory. Both finite and infinite groups are discussed. All the relevant concepts and definitions are introduced.

Lie groups are introduced in chapter 3. In particular, the classical Lie groups are studied in detail. The Haar measure is also discussed and examples are provided.

In chapter 4 Lie transformation groups are defined and a large number of applications are provided.

Chapter 5 is devoted to the infinitesimal transformations (vector fields) of Lie transformation groups. In particular, the three theorems of Lie are discussed.

Chapter 6 gives a comprehensive introduction into Lie algebras. We also discuss representations of Lie algebras in details. Many examples are provided to clarify the definitions and theorems. Also concepts important in theoretical physics such as Casimir operators and Cartan-Weyl basis are provided.

The form-invariance of partial differential equations under Lie transformation groups is illustrated by way of examples in chapter 7. This should be seen as an introduction to the development of the theory of invariance of differential equations by the jet bundle formalism. The Gauge transformation for the Schrödinger equation is also discussed. We also show how the electromagnetic field A_μ is coupled to the wave function ψ.

Chapter 8 deals with differential geometry. This means we consider differential forms and tensor fields. Theorems and definitions (with examples) are provided that are of importance in the application of Lie algebras to differential equations. A comprehensive introduction into differential forms and tensor fields is given.

The Lie derivative is of central importance for continuous symmetries with applications to differential equations. In chapter 9 we study invariance and conformal invariance of geometrical objects, i.e. functions, vector fields, differential forms, tensor fields, etc..

In chapter 10 the jet bundle formalism in connection with the prolongation of vector fields and (partial) differential equations is studied. The application of the Lie derivative in the jet bundle formalism is analysed to obtain the invariant Lie algebra. Explicit analytic solutions are then constructed by applying the invariant Lie algebra. These are the so-called similarity solutions which are of great theoretical and practical importance. The direct method is also introduced.

In chapter 11 the generalisation of the Lie point symmetry vector fields is considered. These generalised vector fields are known as the Lie-Bäcklund symmetry vector fields. Similarity solutions are constructed from the Lie-Bäcklund vector fields. The connection with gauge transformations is also discussed.

In chapter 12 the inverse problem is considered. This means that a partial differential equation is constructed from a given Lie algebra which is spanned by Lie point or Lie-Bäcklund symmetry vector fields.

A list of Lie symmetry vector fields of some important partial differential equations in physics is included in chapter 13. In particular the Lie symmetry vector fields for the Maxwell-Dirac equation have been calculated.

In chapter 14 the Gateaux derivative is defined. A Lie algebra is introduced using the Gateaux derivative. Furthermore, recursion operators are defined and applied. Then we can find hierarchies of integrable equations.

In chapter 15 we introduce auto-Bäcklund transformations and Bäcklund transformations.for partial and ordinary differential equations. We show that these transformations can be used to construt solutions.

For soliton equations the Lax representations are the starting point for the inverse scattering method. In chapter 16 we discuss the Lax representation. Many illustrative examples are given. Sato's theory is also included.

The important concept of conservation laws is discussed in chapter 17. The connection between conservation laws and Lie symmetry vector fields is of particular interest. Extensive use is made of the definitions and theorems of exterior differential forms. The Cartan fundamental form plays an important role regarding the Lagrange density and Hamilton density. String theory and invariants are also discussed.

In chapter 18 the Painlevé test is studied with regard to the symmetries of ordinary and partial differential equations. The Painlevé test provides an approach to study the integrability of ordinary and partial differential equations. This approach is studied and several examples are given. In particular a connection between the singularity manifold and similarity variables is presented. The connection of the Hirota technique and the Painlevé test is also discussed in detail.

Ziglin's theorem can be used to decide whether an ordinary differential equation can be integrated. This theorem is discussed in chapter 19 together we many applications.

In chapter 20 the extension of differential forms, discussed in chapter 7, to Lie algebra valued differential forms is studied. The covariant exterior derivative is defined. Then the Yang-Mills equations and self-dual Yang-

Mills equations are introduced. It is conjectured that the self-dual Yang-Mills equations are the master equations of all integrable equations such as the Korteweg-de Vries equation. The geometry of the Lie group $SU(n)$ is also described in detail.

The connection between nonlinear autonomous systems of ordinary differential equations, first integrals, Bose operators and Lie algebras is studied in chapter 21. It is shown that ordinary differential equations can be expressed with Bose operators. Then the time-evolution can be calculated using the Heisenberg picture. An extension to nonlinear partial differential equations is given where Bose field operators are considered. Difference equations and Bose operators are also investigated.

The concepts of invariants for maps are introduced in chapter 22 with a large number of examples. In particular, the logistic map and the Fibonacci trace map are investigated. The discrete Painlevé equations are also discussed and examples are provided.

Chapter 23 gives a survey of computer algebra packages. Of particular interest are the computer programs available for the calculation of symmetry vector fields. SymbolicC++, a computer algebra package embedded in C++, is introduced and a number of SymbolicC++ programs useful for the different topics in the book are provided.

The appendix provides an introduction to differentiable manifolds.

The emphasis throughout this book is on differential equations and difference equations that are of importance in physics and engineering. The examples and applications consist mainly of the following equations: the Korteweg-de Vries equation, the sine-Gordon equation, Burgers' equation, linear and nonlinear diffusion equations, the Schrödinger equation, the nonlinear Klein-Gordon equation, nonlinear Dirac equations, Yang-Mills equations, the Lorenz model, the Lotka-Volterra model, damped and driven anharmonic oscillators, and differential equations in string theory.

Each chapter includes a section on computer algebra applications with a SymbolicC++ programs. Each chapter also includes exercises.