2 Fractals

Many spatial structures in nature result from the self-assembly of a large number of identical components. To be efficient, the self-assembly process takes advantage of and occurs via some simple prescriptions, which we call the principles of organization. The two simplest principles are the principle of regularity and the principle of randomness. With the former, the components arrange themselves in a periodic or quasiperiodic regular fashion, resulting in crystals, alloys, a formation of soldiers in a parade, etc. Examples of structures (or nonstructures) resulting from the latter are those in gases and the distribution of animal hairs.

Between these two extremes there is the principle of self-similarity, leading to self-similar structures called fractals. In a self-similar fractal, part of the system, when blown up in scale (with the *same* magnification in different directions), resembles the whole. A fractal usually has a fractional dimension. These concepts can be illustrated by the example of a Sierpinski gasket (SG). To construct the SG, in the first step ($n = 0$) let us start with an equilateral triangle with each side equal to one. In the next step ($n = 1$) cut out the middle inverted triangle; in step $n = 2$, do the same for each of the three triangles left over from the previous step (Fig. 2.1). Repeat this cutting procedure until $n = \infty$. (Of course, this can be done only in your mind but not in practice.) The set of "triangles" left at step $n = \infty$ is the Sierpinski gasket. It is easy to see that every small part of the SG has the same shape as the whole; the SG is thus a self-similar fractal. [The SG can be generated by a computer in at least four different ways (Sections 9.6, 13.2, 14.1.1 and 14.1.2).]

The dimension of an object D is given by

$$N_\varepsilon \sim \varepsilon^{-D} \tag{2.1}$$

where N_ε is the minimal number of identical small objects (of linear size ε each) needed to cover the original object. Here the tilde denotes "proportional to when $\varepsilon \to 0$." The dimension D so defined by Eq. (2.1) is called the box dimension

(Section 14.1.3). Note that Eq. (2.1) is equivalent to

$$D = \lim_{\varepsilon \to 0} (-\log N_\varepsilon / \log \varepsilon) \tag{2.2}$$

[That Eq. (2.1) indeed gives $D = 2$ for a square can be checked easily by using small squares to cover up the original square. The procedure is similar to what is shown in Fig. 2.1 below.]

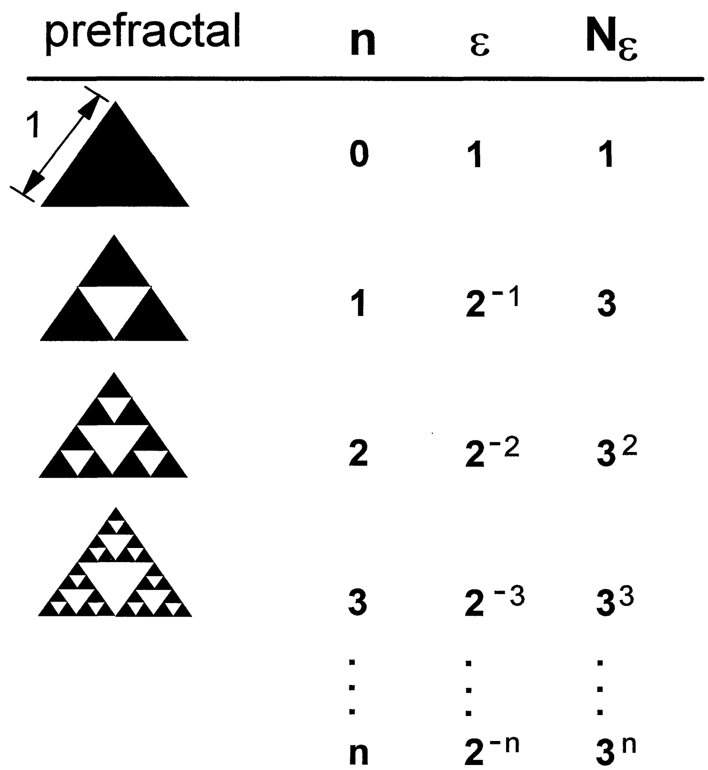

Fig. 2.1. The construction of the Sierpinski gasket, and the procedure in determining its (fractal) dimension. Equilateral triangles of linear size ε are used to cover the gasket. The figure at $n = 0$ is called the initiator; the one at $n = 1$ is called the generator; each figure at step n is called a prefractal.

To determine the dimension of the SG, let us try to cover it with small equilateral triangles. Note that the SG cannot be drawn out explicitly (it exists only at $n = \infty$). However, one can still proceed as shown in Fig. 2.1. For each ε, the number of small triangles N_ε, shown on each row, definitely cover the figure on that row and hence the SG itself, since the SG is a subset of this figure. One therefore has $N_\varepsilon = 3^n$ for $\varepsilon = 1/2^n$. Using Eq. (2.2) and the fact that $\log N_\varepsilon / \log \varepsilon = (n \log 3)/(-n \log 2)$, one obtains $D = \log 3 / \log 2 \approx 1.58$, which is *not* an integer.

For fractals generated from growth processes, one can define the fractal dimension by

$$M \sim R^D \tag{2.3}$$

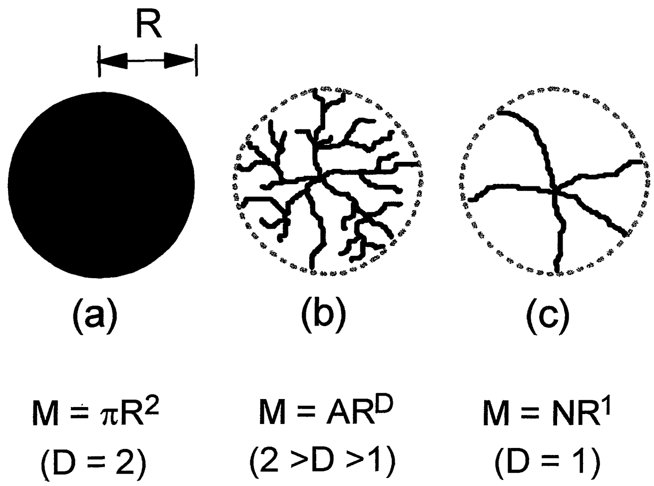

Fig. 2.2. The definition of dimension D and the origin of a fractional dimension in a growth process. M is the mass which is proportional to the black area. R is the radius, the linear size of the object. A in (b) is a constant; N is the number of lines in (c). In (a) the growth object is a solid circle; in (b) a fractal "tree" with branching; and in (c), a simple tree without branching. We thus see that branching is an essential ingredient in the formation of a fractal tree.

where M is the mass of the object when its linear size is R. Here the tilde means "proportional to when $R \to \infty$." The reason that Eq. (2.3) can give rise to a nonintegral value of D is illustrated in Fig. 2.2. [The dimensions obtained from Eqs. (2.1) and (2.3) are usually equal to each other.] Two well-known fractal growth models are the diffusion-limited aggregation model (Sections 9.1 and 14.1.4) and the dielectric breakdown model (Sections 12.2 and 14.1.5). In these models, self-similarity is valid in the statistical sense.

The ubiquitous existence of fractals in natural and mathematical systems became widely known to scientists in the early 1980s after the book *The Fractal Geometry of Nature* by Benoit Mandelbrot was published. Examples of fractals include crumpled paper balls (Section 9.2), aggregates and colloids, trees, rocks, mountains, clouds, galaxies (Section 9.3), polymers, materials with lock-in properties (Section 9.4), fractures (Section 16.3), and the stock market.

When it is required to blow up a part of the object with *different* magnifications in different directions for it to resemble the whole, the object is said to be a self-affine fractal. Interfaces and rough surfaces are such examples (Fig. 2.3). Many fractals are also multifractals, which can be roughly considered as a collection of fractals (Sections 9.5 and 9.6).

A hint of the secret of fractals lies in their power-law behavior, such as the one in Eq. (2.1) or Eq. (2.3). The power law

$$y = Ax^a \tag{2.4}$$

is equivalent to

$$y(\lambda x) = \lambda^a y(x), \quad \text{for } all\ \lambda > 0 \tag{2.5}$$

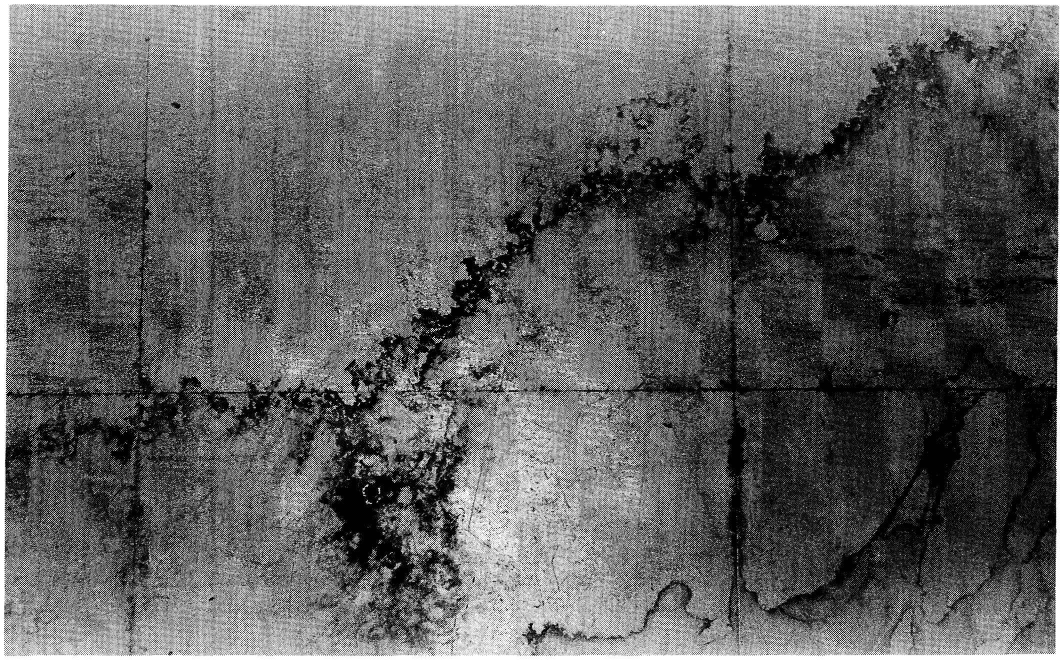

Fig. 2.3. The upper irregular curve is a self-affine fractal. The image resulted from the dried liquid splashed on the floor of the author's laboratory after the liquid underwent a free fall from a leaking pipe at the ceiling of the room.

The fact that Eq. (2.4) implies Eq. (2.5) is established by direct substitution. Since λ is arbitrary, one may choose $\lambda = 1/x$ and Eq. (2.5) reduces to $y(x) = y(1)x^a$, which is Eq. (2.4) with $A = y(1)$. The two equations are thus equivalent to each other. (The positiveness of λ ensures that λ^a is always a real number.)

Mathematically, any function $y(x)$ that satisfies Eq. (2.5) is called a homogeneous function. A homogeneous function is scale invariant, i.e., if we change the scale of measuring x so that $x \to x'$ $(\equiv \lambda x)$, the new function $y'(x')$ $[\equiv y(x)]$ still has the same *form* as the old one $y(x)$. This fact is guaranteed since $y(x) = \lambda^{-a} y(x')$ by Eq. (2.5), and hence $y'(x') \sim y(x')$.

Scale invariance means that if a part of a system is magnified to the size of the original system, this magnified part and the original system will look similar to each other. In other words, there is no intrinsic scale in the original system. A scale-invariant system must be self-similar and vice versa.

Thus we see that self-similarity, spatial power laws and scale invariance are three equivalent ways of expressing the fact that the system lacks a characteristic length scale. Similarly, the absence of a characteristic time scale in the system leads to temporal power laws (e.g., the $1/f$ noise, another ubiquitous phenomenon in nature). It must be noted that power laws are nonlinear equations except when the exponent is unity. To explain the widespread existence of fractals and scale-free behaviors in nonequilibrium systems, the hypothesis of self-organized criticality was proposed by Per Bak, Chao Tang and Kurt Wiesenfeld in 1987, which is supposed to be applicable to sandpiles and many other natural and social systems (Section 13.3).

Lastly, let us note that the fractal concept has found applications in the social sciences, as evidenced by Hans-Jürgan Warnecken's book *The Fractal Company: A Revolution in Corporate Culture* (Springer, 1993).