DEFORMATION QUANTISATION AND CONNECTIONS

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After a brief introduction to the concept of formal Deformation Quantisation, we shall focus on general konwn constructions of star products, enhancing links with linear connections.

We first consider the symplectic context: we recall how any natural star product on a symplectic manifold determines a unique symplectic connection and we recall Fedosov's construction which yields a star product, given a symplectic connection.

In the more general context, we consider universal star products, which are defined by bidifferential operators expressed by universal formulas for any choice of a linear torsionfree connection and of a Poisson structure. We recall how formality implies the existence (and classification) of star products on a Poisson manifold. We present Kontsevich formality on $\mathbb{R}^d$ and we recall how Cattaneo-Felder-Tomassini globalisation of this result proves the existence of a universal star product.

Keywords: Deformation quantisation; star product; symplectic connection; universal star product.

1. Quantization

Quantisation of a classical system is a way to pass from classical to quantum results.

Classical mechanics is considered in its Hamiltonian formulation on the motion space, which is the quotient of the evolution space (usually the product of the phase space and the real line) by the trajectories. Thus the framework is a symplectic manifold (or, more generally, when one deals with constraints, a Poisson manifold).

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http://www.worldscibooks.com/mathematics/7352.html

In this setting a point represents a motion, so that an observable is represented by a family of smooth functions on that manifold M. The dynamics is defined in terms of a Hamiltonian $H \in C^\infty(M)$ and the time evolution of an observable $f_t \in C^\infty(M \times \mathbb{R})$ is governed by the equation:

$$\frac{d}{dt} f_t = -\{H, f_t\}.$$

Quantum mechanics is considered in its usual Heisenberg's formulation. The framework is a Hilbert space (states are rays in that space). An observables is described by a one parameter family of selfadjoint operators on that Hilbert space. The dynamics is defined in terms of a Hamiltonian H, which is a selfadjoint operator, and the time evolution of an observable A_t is governed by the equation:

$$\frac{dA_t}{dt} = \frac{i}{\hbar}[H, A_t].$$

A natural suggestion for quantisation is a correspondence $Q\colon f \mapsto Q(f)$ mapping a function f to a self adjoint operator $Q(f)$ on a Hilbert space $\mathcal{H}$ in such a way that $Q(1) = \mathrm{Id}$ and

$$[Q(f), Q(g)] = i\hbar Q(\{f, g\}).$$

There is no such correspondence defined on all smooth functions on M when one puts an irreducibility requirement which is necessary not to violate Heisenberg's principle.

To deal with this problem, various mathematical theories of quantization were proposed. Deformation Quantisation was introduced in the seventies by Flato, Lichnerowicz and Sternheimer in [10] and developed in [3]; they "suggest that quantisation be understood as a deformation of the structure of the algebra of classical observables rather than a radical change in the nature of the observables."

One stresses here the fundamental aspect of the space of observables rather than the set of states; observables behave indeed in a nicer way when one deels with composed systems: both in the classical and in the quantum picture, the space of observables for combined systems is the tensor product of the spaces of observables.

The algebraic structure of classical observables that one deforms is the algebraic structure of the space of smooth functions on a Poisson manifold: the associative structure given by the usual product of functions and the Lie structure given by the Poisson bracket. Formal deformation quantisation is defined in terms of a formal deformation of that structure called a star product.

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2. Basic definitions

Definition 2.1. A **Poisson bracket** defined on the space of smooth functions on a manifold M, is a $\mathbb{R}$- bilinear skewsymmetric map:

$$C^\infty(M) \times C^\infty(M) \to C^\infty(M) \quad (u, v) \mapsto \{u, v\}$$

satisfying Jacobi's identity ($\{\{u, v\}, w\} + \{\{v, w\}, u\} + \{\{w, u\}, v\} = 0$) and Leibniz rule ($\{u, vw\} = \{u, v\}w + \{u, w\}v \ \ \forall u, v, w \in C^\infty(M)$).

Leibniz rule says that bracketing with a given function u is a derivation of the associative algebra of smooth functions on M, hence is given by a vector field X_u on M. By skewsymmetry, a Poisson bracket is thus given in terms of a contravariant skew symmetric 2-tensor P on M, called the **Poisson tensor**, by

$$\{u, v\} = P(du \wedge dv). \tag{1}$$

The Jacobi identity for the Poisson bracket Lie algebra is equivalent to the vanishing of the Schouten bracket:

$$[P, P] = 0.$$

(The Schouten bracket is the extension -as a graded derivation for the exterior product- of the bracket of vector fields to skewsymmetric contravariant tensor fields; it will be developed further in section 4.1.)

A **Poisson manifold**, denoted (M, P), is a manifold M with a Poisson bracket defined by the Poisson tensor P.

A particular class of Poisson manifolds, essential in classical mechanics, is the class of symplectic manifolds.

Definition 2.2. A **symplectic manifold**, denoted by (M, ω), is a manifold M endowed with is a closed nondegenerate 2-form ω. The corresponding Poisson bracket of two functions $u, v \in C^\infty(M)$ is defined by

$$\{u, v\} := X_u(v) = \omega(X_v, X_u), \tag{2}$$

where X_u denotes the **Hamiltonian vector field** corresponding to the function u, i.e. such that $i(X_u)\omega = du$. In coordinates, the components of the corresponding Poisson tensor P^{ij} form the inverse matrix of the components ω_{ij} of ω.

Examples of symplectic manifolds are given by **cotangent bundles**; if N is a manifold, its cotangent bundle $T^*N \xrightarrow{\pi} N$ is endowed with a canonical 1-form λ defined by

$$\lambda_\xi(Y) = \langle \xi, \pi_* Y \rangle \qquad \xi \in T^*N, \ Y \in T_\xi(T^*N),$$

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where $\langle\cdot,\cdot\rangle$ denotes the pairing between the tangent space T_xN at a point $x=\pi(\xi)\in N$ and its dual space, the cotangent space at x, T^*_xN.
Then $(T^*N, d\lambda)$ is a symplectic manifold.

Duals of Lie algebras form the class of linear Poisson manifolds. If $\mathfrak{g}$ is a Lie algebra then its dual $\mathfrak{g}^*$ is endowed with the Poisson tensor P defined by

$$P_\xi(X,Y) := \xi([X,Y])$$

for $X, Y \in \mathfrak{g} = (\mathfrak{g}^*)^* \sim (T_\xi\mathfrak{g}^*)^*$.

Definition 2.3. [3] A **star product** on (M,P) is a bilinear map

$$C^\infty(M)\times C^\infty(M) \to C^\infty(M)[\![\nu]\!] \qquad (u,v)\mapsto u*v=\sum_{r\geq 0}\nu^r C_r(u,v)$$

such that

(1) extending the map $\mathbb{R}[\![\nu]\!]$-bilinearly to $C^\infty(M)[\![\nu]\!]\times C^\infty(M)[\![\nu]\!]$, it is formally associative:

$$(u*v)*w = u*(v*w);$$

(2) (a) $C_0(u,v)=uv$, (b) $C_1(u,v)-C_1(v,u)=\{u,v\}$;
(3) $1*u=u*1=u$.

When the C_r's are bidifferential operators on M, one speaks of a **differential star product**. When, furthermore, each bidifferential operator C_r is of order maximum r in each argument, one speaks of a **natural star product**.

Given any star product $*$ on (M,P) and any series $T=\mathrm{Id}+\sum_{r=1}^\infty \nu^r T_r$ of linear operators on $C^\infty(M)$, one can build a new star product $*'$ defined by $f*'g=T(T^{-1}f*T^{-1}g)$. Remark that any such series T can be written in the form $T=\operatorname{Exp}E$ where $E=\sum_{r=1}^\infty \nu^r E_r$ with the E_r linear operators on $C^\infty(M)$ and where Exp denotes the exponential series. This motivates the following definition of equivalence:

Definition 2.4. Two star products $*$ and $*'$ on (M,P) are **equivalent** if and only if there is a series

$$E=\sum_{r=1}^\infty \nu^r E_r$$

where the E_r are linear operators on $C^\infty(M)$, such that

$$f*'g=\operatorname{Exp}E\,((\operatorname{Exp}-E)\,f*(\operatorname{Exp}-E)\,g)). \tag{3}$$

Remark 2.1. Two differential star products $*$ and $*'$ on (M,P) are equivalent iff there is a series $E=\sum_{r=1}^{\infty}\nu^r E_r$ where the E_r are differential operators , giving the equivalence [i.e. such that $f *' g = \operatorname{Exp} E\,((\operatorname{Exp} -E)\, f * (\operatorname{Exp} -E)\, g))$].

Two natural star products $*$ and $*'$ on (M,P) are equivalent iff there is a series $E=\sum_{r=1}^{\infty}\nu^r E_r$ where the E_r are differential operators of order at most $r+1$, giving the equivalence.

An example: Moyal star product on $\mathbb{R}^n$

Consider a vector space $V=\mathbb{R}^m$ with a Poisson structure P with constant coefficients:

$$P=\sum_{i,j} P^{ij}\partial_i\wedge\partial_j,\;\; P^{ij}=-P^{ji}\in\mathbb{R}$$

where $\partial_i=\partial/\partial x^i$ is the partial derivative in the direction of the coordinate $x^i,\;\; i=1,\ldots,n$. **Moyal star product** is defined by

$$(u *_M v)(z) = \exp\left(\frac{\nu}{2}\sum_{r,s}P^{rs}\partial_{x^r}\partial_{y^s}\right)(u(x)v(y))\Big|_{x=y=z}. \qquad (4)$$

Associativity follows from the fact that

$$\partial_{t^k}(u *_M v)(t) = (\partial_{x^k}+\partial_{y^k})\exp\left(\frac{\nu}{2}P^{rs}\partial_{x^r}\partial_{y^s}\right)(u(x)v(y))\Big|_{x=y=t},$$

so that

$$\begin{aligned}
((u *_M v) *_M w)(x') &= \exp\left(\frac{\nu}{2}P^{rs}\partial_{t^r}\partial_{z^s}\right)((u *_M v)(t)w(z))\Big|_{t=z=x'}\\
&= \exp\left(\frac{\nu}{2}P^{rs}(\partial_{x^r}+\partial_{y^r})\partial_{z^s}\right)\exp\left(\frac{\nu}{2}P^{ab}\partial_{x^a}\partial_{y^b}\right)(u(x)v(y))w(z))\Big|_{x=y=z=x'}\\
&= \exp\left(\frac{\nu}{2}P^{rs}(\partial_{x^r}\partial_{z^s}+\partial_{y^r}\partial_{z^s}+\partial_{x^r}\partial_{y^s})\right)(u(x)v(y))w(z))\Big|_{x=y=z=x'}\\
&= (u *_M (v *_M w)(x').
\end{aligned}$$

Remark that one can define by an analogous formula a Moyal star product on a Poisson manifold (M,P) as soon as the Poisson tensor writes

$$P=\sum_{i,j} P^{ij}X_i\wedge X_j,\;\; P^{ij}=-P^{ji}\in\mathbb{R}$$

where the X_i's form a set of commuting vector fields on M.

Definition 2.5. When the constant Poisson structure P is non degenerate (which implies $V=\mathbb{R}^{2n}$), the space of polynomials in ν whose coefficients are polynomials on V with Moyal product is called **the Weyl algebra** $(S(V^*)[\nu], *_M)$.

Moyal star product on $(\mathbb{R}^{2n}, \omega = \sum_i dp_i \wedge dq_i)$ is related to the composition of operators via Weyl's quantisation. This associates to a polynomial f on $\mathbb{R}^{2n}$ a differential operator $Q_W(f)$ on $\mathbb{R}^n$ in the following way: to the classical observables q^i and p_i, one associates the quantum operators $Q^i = q^i\cdot$ and $P_i = -i\hbar\frac{\partial}{\partial q^i}$ acting on functions depending on q^j's. Since Q^i and P_j do no longer commute, one has to specify which operator is associated to a higher degree polynomial in q^i and p_j. The *Weyl ordering* associates the corresponding totally symmetrized polynomial in Q^i and P_j, e.g.

$$Q_W(q^1(p^1)^2) = \frac{1}{3}(Q^1(P^1)^2 + P^1Q^1P^1 + (P^1)^2Q^1).$$

Then

$$Q_W(f) \circ Q_W(g) = Q_W(f *_M g) \quad (\text{for } \nu = i\hbar).$$

Remark that another ordering, such as the standard ordering, which associates to a polynomial f in q^i and p_j the operator $Q_{std}(f)$ with all the Q^i's on the left and the P^j's on the right, gives another isomorphism between the space of differential operators on $\mathbb{R}^n$ and the space of polynomials on $\mathbb{R}^{2n}$. This yields another star product $*_{std}$ on $\mathbb{R}^{2n}$ so that

$$Q_{std}(f) \circ Q_{std}(g) = Q_{std}(f *_{std} g) \quad (\text{for } \nu = i\hbar).$$

One can show that $Q_W(f) = Q_{std}(\exp \frac{i\hbar}{2} Df)$ where $D = \frac{\partial^2}{\partial p \partial q}$ so that

$$\exp \nu D(f *_M g) = \exp \nu D(f) *_{std} \exp \nu D(g).$$

One can show more generally that any two differential star products on $\mathbb{R}^{2n}$ are equivalent.

3. Symplectic case: star products and symplectic connections

A linear connection on a manifold M is a way to differentiate a vector field along a vector field:

Definition 3.1. Let $\chi(M) = \Gamma(TM)$ be the space of smooth vector fields on M. A **linear connection** on M is a bilinear map

$$\nabla : \chi(M) \times \chi(M) \to \chi(M) \quad (X, Y) \mapsto \nabla_X Y$$

such that $\nabla_{fX} Y = f\nabla_X Y$ and $\nabla_X fY = X(f)\, Y + f\nabla_X Y$, $\forall X, Y \in \chi(M)$ and $\forall f \in C^\infty(M)$. Equivalently, it is a $C^\infty(M)$-linear map

$$\nabla : \Gamma(TM) \to \Gamma(T^*M \otimes TM) \quad Y \mapsto \nabla Y$$

so that $\nabla fY = df \otimes Y + f\nabla Y$.

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The **torsion** of a linear connection is the $\binom{1}{2}$-tensor T^∇ on M defined by

$$T^\nabla(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y],$$

and its curvature is the $\binom{1}{3}$-tensor R^∇ defined by

$$R^\nabla(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

A linear connection gives a way to differentiate any tensor field on M along a vector field X: for a smooth function f, one defines $\nabla_X f = Xf$ and for a covariant p-tensor field α one defines

$$(\nabla_X\alpha)(Y_1,\ldots,Y_p) = X(\alpha(Y_1,\ldots,Y_p)) - \sum_{i=1}^{p}\alpha(Y_1,\ldots,\nabla_X Y_i,\ldots,Y_p).$$

Multidifferential operators on a manifold can be written in a global manner through the use of a linear connection : given a torsionfree linear connection ∇ on a manifold M (torsionfree meaning that $T^\nabla = 0$), any multidifferential operator $\mathrm{Op} : (C^\infty(M))^k \to C^\infty(M)$ writes in a unique way as

$$\mathrm{Op}(f_1,\ldots,f_k) = \sum_{J_1,\ldots,J_k} \mathrm{Op}^{J_1,\ldots,J_k}\nabla^{sym}_{J_1} f_1 \ldots \nabla^{sym}_{J_k} f_k \qquad (5)$$

where the $J_1,\ldots,J_k$ are multiindices and $\nabla^{sym}_J f$ is the symmetrised covariant derivative of order $|J|$ of f:

$$\nabla^{sym}_J f = \sum_{\sigma\in S_m}\frac{1}{m!}\nabla^m_{i_{\sigma(1)}\ldots i_{\sigma(m)}} f \qquad \text{for } \mathrm{J} = (\mathrm{i}_1,\ldots,\mathrm{i}_\mathrm{m}),$$

where $\nabla^m_{i_1\ldots i_m} f := \nabla^m f(\partial_{i_1},\ldots,\partial_{i_m})$ with $\nabla^m f$ defined inductively by $\nabla f := df$ and $\nabla^m f(X_1,\ldots,X_m) = (\nabla_{X_1}(\nabla^{m-1}f))(X_2,\ldots,X_m)$.
The tensors $\mathrm{Op}^{J_1,\ldots,J_k}$ are covariant tensors of order $|J_1|+\ldots+|J_k|$ which are symmetric within each block of J_r indices; they are called **the tensors associated** to Op for the given connection.

Star products in the symplectic context are strongly linked to symplectic connections.

Definition 3.2. A **symplectic connection** on a symplectic manifold (M,ω) is a torsionfree linear connection ∇ which is torsionfree and such that the symplectic form ω is parallel, $\nabla\omega = 0$ (i.e. $X(\omega(Y,Z)) = \omega(\nabla_X Y, Z) + \omega(Y, \nabla_X Z)$).

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It is well known that such connections exist; to see this take ∇^0 to be any torsionfree linear connection (for instance, the Levi Civita connection associated to a metric g on M), and define

$$\nabla_X Y := \nabla^0_X Y + \frac{1}{3}N(X,Y) + \frac{1}{3}N(Y,X).$$

where $\nabla^0_X \omega(Y,Z) =: \omega(N(X,Y),Z)$. Then ∇ is symplectic.

Unlike in the Riemann case, symplectic connections are not unique. Take ∇ symplectic; then $\nabla'_X Y := \nabla_X Y + S(X,Y)$ is symplectic if and only if $\omega(S(X,Y),Z)$ is totally symmetric, so the set of symplectic connections is an affine space modelled on the space of contravariant symmetric 3-tensor fields on M.

A first result concerning the link between a star product and a connection is the observation in 1978, in the seminal paper about deformation quantisation [3] by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer that Moyal star product can be defined on any symplectic manifold (M,ω) which admits a symplectic connection ∇ with no curvature.

Lichnerowicz [13] showed that some star products on a symplectic manifold determine a unique symplecic connection; this we generalised as follows

Proposition 3.1. *[11] A star product* $* = \sum_{r\geq 0} \nu^r C_r$ *at order 2 (i.e. satisfying associativity up to terms in* ν^3 *) on a symplectic manifold* (M,ω)*, such that* C_1 *is a bidifferential operator of order* 1 *in each argument and* C_2 *of order at most* 2 *in each argument, determines a unique symplectic connection* $\nabla = \nabla(*)$ *such that*

$$C_1 = \{\ ,\ \} + \mathrm{ad}\, E_1\, m \quad C_2 = \frac{1}{2}(\mathrm{ad}\, E_1)^2\, m + ((\mathrm{ad}\, E_1)\,\{\ ,\ \}) + \frac{1}{2}P^2(\nabla^2\cdot,\nabla^2\cdot) + A_2 \qquad (6)$$

where m *is the usual multiplication of functions, where*

$$(\mathrm{ad}\, E\, C)(u,v) = E(C(u,v)) - C(Eu,v) - C(u,Ev)$$

for any 1*-diefferential operator* E *and any bidifferential operaotor* C*, where* A_2 *is a skewsymmetric bidifferential operator of order* 1 *in each argument, and where* $P^2(\nabla^2\cdot,\nabla^2\cdot)$ *denotes the bidifferential operator which is given by* $\sum_{ij} P^{ij} P^{i'j'}\, \nabla^2_{ii'} u\, \nabla^2_{jj'} v$ *in a chart.*

In particular, any natural star product $* = \sum_{r\geq 0} \nu^r C_r$ *on a symplectic manifold* (M,ω) *determines a unique symplectic connection.*

Reciprocally, Fedosov gave in 1985 (but appearing only in the West in the nineties [9]), a recursive construction of a star product on a symplectic manifold endowed with a symplectic connection. The first proof of the existence of a star product on any symplectic manifold had been given in 1983 by De Wilde and Lecomte [7]; this was obtained by building at the same time the star product and a special derivation of it. Fedosov's construction yields a star product such that the bidifferential operators defining it are given by universal formulas in terms of the symplectic 2-form, the curvature of the connection and all its covariant derivatives.

3.1. *Fedosov's construction*

Fedosov's construction [9] gives a star product on a symplectic manifold (M, ω), when one has chosen a symplectic connection and a sequence of closed 2-forms on M. The star product is obtained by identifying the space $C^\infty(M)[[\nu]]$ with an algebra of flat sections of the so-called Weyl bundle endowed with a flat connection.

Definition 3.3. Let (V, Ω) be a symplectic vector space. We endow the space of polynomials in ν whose coefficients are polynomials on V with Moyal star product (this is the Weyl algebra $S(V^*)[\nu]$). This algebra is isomorphic to the universal enveloping algebra of the Heisenberg Lie algebra $\mathfrak{h} = V^* \oplus \mathbb{R}\nu$ with Lie bracket

$$\left[y^i, y^j\right] = (\Omega^{-1})^{ij}\nu.$$

[Indeed both associative algebras $U(\mathfrak{h})$ and $S(V^*)[\nu]$ are generated by V^* and ν and the map sending an element of $V^* \subset \mathfrak{h}$ to the corresponding element in $V^* \subset S(V^*)$ viewed as a linear function on V and mapping $\nu \in \mathfrak{h}$ on $\nu \in \mathbb{R}[\nu] \subset S(V^*)[\nu]$ has the universal property:

$$\xi *_M \xi' - \xi' *_M \xi = [\xi, \xi'] \quad \forall \xi, \xi' \in \mathfrak{h} = V^* \oplus \mathbb{R}\nu$$

so extends to a morphism of associative algebras.]

One defines a grading on $U(\mathfrak{h})$ assigning the degree 1 to each $y^i \in V^*$ and the degree 2 to ν. The **formal Weyl algebra** W is the completion in that grading of the above algebra. An element of the formal Weyl algebra is of the form

$$a(y, \nu) = \sum_{m=0}^{\infty} \left(\sum_{2k+l=m} a_{k,i_1,\ldots,i_l} \nu^k y^{i_1} \ldots y^{i_l} \right).$$

The product in $U(\mathfrak{h})$ is given by the Moyal star product and is extended to W:

$$(a \circ b)(y, \nu) = \left(\exp\left(\frac{\nu}{2} P^{ij} \frac{\partial}{\partial y^i} \frac{\partial}{\partial z^j} \right) a(y, \nu) b(z, \nu) \right)\Bigg|_{y=z}$$

with $P^{ij} = (\Omega^{-1})^{ij}$.

The symplectic group

$$Sp(V,\Omega) := \{A : V \to V \text{ linear} \,|\, \Omega(Au, Av) = \Omega(u, v)\, \forall u, v \in V\}$$

acts as automorphisms of $\mathfrak{h}$ by $A \cdot f = f \circ A^{-1}$ for $f \in V^*$ and $A \cdot \nu = 0$, and this action extends to both $U(\mathfrak{h})$ and W; on the latter we denote it by ρ. It respects the multiplication $\rho(A)(a \circ b) = \rho(A)(a) \circ \rho(A)(b)$. Explicitly, we have:

$$\rho(A)(\sum_{2k+l=m} a_{k,i_1,\ldots,i_l}\nu^k y^{i_1}..y^{i_l}) = \sum_{2k+l=m} a_{k,i_1,\ldots,i_l}\nu^k (A^{-1})^{i_1}_{j_1}..(A^{-1})^{i_l}_{j_l} y^{j_1}..y^{j_l}.$$

To an element B in the Lie algebra $sp(V,\Omega)$ we associate the quadratic element $\overline{B} = \frac{1}{2}\sum_{ijr}\Omega_{ri}B^r_j y^i y^j \in W$. The natural action $\rho_*(B)$ is given by: $\rho_*(B)y^l = \frac{-1}{\nu}[\overline{B}, y^l]$ where $[a,b] := (a \circ b) - (b \circ a)$ for any $a, b \in W$. Since both sides act as derivations this extends to all of W as

$$\rho_*(B)a = \frac{-1}{\nu}[\overline{B}, a]. \qquad (7)$$

Definition 3.4. If (M,ω) is a symplectic manifold, we can form its bundle $F_{symp}(M)$ of symplectic frames. A **symplectic frame** at the point $x \in M$ is a linear symplectic isomorphism $\xi_x : (V,\Omega) \to (T_xM, \omega_x)$. The bundle $F_{symp}(M)$ is a principal $Sp(V,\Omega)$-bundle over M (the action on the right of an element $A \in Sp(V,\Omega)$ on a frame ξ_x is given by $\xi_x \circ A$).

The associated bundle $\mathcal{W} = F_{symp}(M) \times_{Sp(V,\Omega),\rho} W$ is a bundle of algebras on M called the bundle of formal Weyl algebras, or, more simply, **the Weyl bundle**. Its sections have the form of formal series

$$a(x,y,\nu) = \sum_{2k+l\geq 0} \nu^k a_{k,i_1,\ldots,i_l}(x) y^{i_1} \cdots y^{i_l} \qquad (8)$$

where the coefficients $a_{k,i_1,\ldots,i_l}$ define (in the $i's$) symmetric covariant l–tensor fields on M. We denote by $\Gamma(\mathcal{W})$ the space of those sections.

The product of two sections taken pointwise makes $\Gamma(\mathcal{W})$ into an algebra with **multiplication**

$$(a \circ b)(x,y,\nu) = \left(\exp\left(\frac{\nu}{2}\Lambda^{ij}\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j}\right) a(x,y,\nu) b(x,z,\nu)\right)\bigg|_{y=z}. \qquad (9)$$

The center of this algebra coincide with $C^\infty(M)[[\nu]]$.

A symplectic connection defines a connection 1-form in the symplectic frame bundle and so a connection in all associated bundles (i.e. a covariant derivative of sections). In particular we obtain a connection in $\mathcal{W}$ which we denote by ∂; it can be viewed as a map $\partial\colon \Gamma(\mathcal{W}) \to \Gamma(\mathcal{W} \otimes \Lambda^1 T^*M)$ where sections of the bundle $\mathcal{W} \otimes \Lambda^\bullet T^*M$ are $\mathcal{W}$-valued forms on M and have locally the form

$$a = \sum_{p\geq 0, q\geq 0} a_{pq} = \sum_{2k+p\geq 0, q\geq 0} \nu^k a_{k,i_1,\dots,i_p,j_1,\dots,j_q} y^{i_1} \dots y^{i_p}\, dx^{j_1} \wedge \dots \wedge dx^{j_q}$$

with coefficients which are again covariant tensors, symmetric in $i_1, \dots, i_p$ and anti-symmetric in $j_1, \dots, j_q$. [In particular $a_{00} = \sum_k \nu^k a_k$ with $a_k \in C^\infty(M)$.] Such sections can be multiplied using the product in $\mathcal{W}$ and simultaneously exterior multiplication $a \otimes \omega \circ b \otimes \omega' = (a \circ b) \otimes (\omega \wedge \omega')$ and bracketed

$$[s, s'] = s \circ s' - (-1)^{q_1 q_2} s' \circ s$$

if $s_i \in \Gamma(\mathcal{W} \otimes \Lambda^{q_i} T^*M)$.

Let Γ^i_{kl} be the Christoffel symbols of the chosen symplectic connection ∇ and let $\overline{\Gamma} := \frac{1}{2} \sum_{ijkr} \omega_{ki} \Gamma^k_{rj} y^i y^j \, dx^r$; then the connection in $\mathcal{W}$ is given by

$$\partial a = da - \frac{1}{\nu}[\overline{\Gamma}, a].$$

For any vector field X on M, the covariant derivative ∂_X is a derivation of the algebra $\Gamma(\mathcal{W})$.

As usual, the connection ∂ in $\mathcal{W}$ extends to a covariant exterior derivative on all of $\Gamma(\mathcal{W} \otimes \Lambda^\bullet T^*M)$, also denoted by ∂, by using the Leibniz rule:

$$\partial(a \otimes \omega) = \partial(a) \wedge \omega + a \otimes d\omega.$$

The curvature of ∂ is then given by $\partial_\circ \partial$ which is a 2-form with values in $\mathrm{End}(\mathcal{W})$. If R denotes the curvature of the symplectic connection ∇:

$$\partial_\circ \partial a = \frac{1}{\nu}[\overline{R}, a]$$

where $\overline{R} = \frac{1}{4} \sum_{ijklr} \omega_{rl} R^l_{ijk} y^r y^k \, dx^i \wedge dx^j$.

The idea is to try to modify ∂ to have zero curvature; we look for a connection D on $\mathcal{W}$, so that D_X is a derivation af the algebra $\Gamma(\mathcal{W})$ for any vectorfield X on M, and so that D is flat in the sense that $D_\circ D = 0$. Such a connection can be written as a sum of ∂ and a $\mathrm{End}(\mathcal{W})$-valued 1-form. The latter is taken in a particular form:

$$Da = \partial a - \delta(a) - \frac{1}{\nu}[r, a] \tag{10}$$

with $\delta(a) = \sum_k dx^k \wedge \frac{\partial a}{\partial y^k} = \frac{1}{\nu}\left[\sum_{ij} -\omega_{ij} y^i dx^j, a\right]$, so that $\delta^2 = 0$, $\partial\delta + \delta\partial = 0$ and δ is a graded derivation of $\Gamma(\mathcal{W} \otimes \Lambda^\bullet T^*M)$. Then

$$D_\circ Da = \frac{1}{\nu}\left[\overline{R} - \partial r + \delta r + \frac{1}{2\nu}[r,r], a\right]$$

and $[r,r] = 2r \circ r$. So we will have a flat connection D provided we can make the first term in the bracket be a central 2-form. Introducing

$$\delta^{-1}(a_{pq}) = \begin{cases} \frac{1}{p+q}\sum_k y^k i(\frac{\partial}{\partial x^k})a_{pq} \text{ if } \ \mathrm{p+q} > 0; \\ 0 \text{ if } \ \mathrm{p+q} = 0. \end{cases}$$

Then $(\delta^{-1})^2 = 0$ and $(\delta\delta^{-1} + \delta^{-1}\delta)(a) = a - a_{00}$.

Theorem 3.1. *(Fedosov [9]) The equation*

$$\delta r = -\overline{R} + \partial r - \frac{1}{\nu} r^2 + \tilde{\Omega} \tag{11}$$

for a given formal series $\tilde{\Omega} = \sum_{i \geq 1} h^i \omega_i$ of closed 2-forms ω_i on M, has a unique solution $r \in \Gamma(\mathcal{W} \otimes \Lambda^1 T^*M)$ satisfying the normalization condition $\delta^{-1} r = 0$ and such that the $\mathcal{W}$-degree of the leading term of r is at least 3. It is inductively defined by

$$r = -\delta^{-1}\overline{R} + \delta^{-1}\partial r - \frac{1}{\nu}\delta^{-1} r^2 + \delta^{-1}\tilde{\Omega}. \tag{12}$$

Since D_X acts as a derivation of the pointwise multiplication of sections, the space $\mathcal{W}_D$ of flat sections will be a subalgebra of the space of sections of $\mathcal{W}$:

$$\mathcal{W}_D = \{a \in \Gamma(\mathcal{W}) | Da = 0\}.$$

The importance of this space of sections comes from the fact that there is a bijection between this space $\mathcal{W}_D$ and the space of formal power series of smooth functions on M.

Theorem 3.2. *[9] Given a flat connection D, for any $a_\circ \in C^\infty(M)[[\nu]]$ there is a unique $a \in \mathcal{W}_D$ such that $a(x, 0, \nu) = a_\circ(x, \nu)$. It is defined inductively by*

$$a = \delta^{-1}\delta a + a_\circ = \delta^{-1}\left(\partial a - \frac{1}{\nu}[r, a]\right) + a_\circ. \tag{13}$$

One defines **the symbol map** $\sigma : \Gamma(\mathcal{W}) \to C^\infty(M)[[\nu]]$, by $\sigma(a) = a(x, 0, \nu)$. Theorem 3.2 tells us that σ is a linear isomorphism when restricted to $\mathcal{W}_D$. So it can be used to transport the algebra structure of $\mathcal{W}_D$ to $C^\infty(M)[[\nu]]$. **Fedosov's star product** is defined by :

$$a *_F^{\nabla,\Omega} b = \sigma(\sigma^{-1}(a) \circ \sigma^{-1}(b)), \quad a, b \in C^\infty(M)[[\nu]]. \tag{14}$$

Remark that its construction depends only on the choice of a symplectic connection ∇ and the choice of a series Ω of closed 2-forms on M. If the curvature and the Ω vanish, one gets back the Moyal $*$-product.

4. Star products on Poisson manifolds

In the Poisson context, generally speaking one cannot find a "Poisson connection". Indeed, if one looks for a linear connection such that $\nabla P = 0$, then the rank of the Poisson structure must be constant. So the best we can do in general is to consider a torsionfree linear connection.

We consider star products on a manifold M which are given by universal formulas when one has chosen any Poisson structure and any connection on M. By universal, we mean the following:

Definition 4.1. [2] A **universal star product** $* = m + \sum_{r\geq 1} \nu^r C_r$ will be the association to any manifold M, any torsionfree connection ∇ on M and any Poisson tensor P on M, of a differential star product

$$*^{(M,\nabla,P)} := m + \sum_{r\geq 1} \nu^r C_r^{(M,\nabla,P)}$$

where each C_r is a universal Poisson-related bidifferential operator, i.e. so that, the tensors associated to $C_r^{(M,\nabla,P)}$ for ∇ are given by universal polynomials, involving concatenations, in P, the curvature tensor R and their covariant multiderivatives.

Kontsevich proved in 1997 the existence of a star product on any Poisson manifold as a consequence of his formality theorem. He gave an explicit formula for a formality and thus for a star product on $\mathbb{R}^d$ endowed with any Poisson structure. We shall now present this result.

4.1. *Star products on Poisson manifolds and formality*

Kontsevich in [12] showed that the set of equivalence classes of star products is the same as the set of equivalence classes of formal Poisson structure. A differential star product on M is defined by a series of bidifferential operators satisfying some identities; a formal Poisson structure on a manifold M

is completely defined by a series of bivector fields P satisfying certain properties. To describe a correspondence between these objects, one introduces the algebras they belong to.

Definition 4.2. A **differential graded Lie algebra** (briefly DGLA) is a $\mathbb{Z}$-graded vector space $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}^i$ endowed with
• a structure of graded Lie algebra, i.e. a graded bilinear map

$$[\ ,\ ] \colon \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g} \text{ such that } [\,a\,,\,b\,] \subset \mathfrak{g}^{\alpha+\beta}$$

which is graded skewsymmetric ($[\,a\,,\,b\,] = -(-1)^{\alpha\beta}[\,b\,,\,a\,]$) and which satisfies the graded Jacobi identities: $[\,a\,,[\,b\,,\,c\,]] = [[\,a\,,\,b\,],\,c\,] + (-1)^{\alpha\beta}[\,b\,,[\,a\,,\,c\,]]$ for any $a\in\mathfrak{g}^\alpha$, $b\in\mathfrak{g}^\beta$ and $c\in\mathfrak{g}^\gamma$,
• together with a differential, $d\colon \mathfrak{g}\to\mathfrak{g}$, i.e. a linear operator of degree 1 ($d\colon \mathfrak{g}^i \to \mathfrak{g}^{i+1}$) which squares to zero ($d\circ d = 0$)
• satisfying the compatibility condition (Leibniz rule)

$$d[\,a\,,\,b\,] = [\,d\,a\,,\,b\,] + (-1)^\alpha[\,a\,,\,d\,b\,] \qquad\qquad a\in\mathfrak{g}^\alpha, b\in\mathfrak{g}^\beta.$$

Star products and the DGLA of polydifferential operators

Let A be an associative algebra with unit on a field $\mathbb{K}$; consider the complex of multilinear maps from A to itself:

$$\mathcal{C} := \sum_{i=-1}^{\infty} \mathcal{C}^i \qquad \mathcal{C}^i := \mathrm{Hom}_{\mathbb{K}}(A^{\otimes(i+1)}, A)$$

remark that the degree $|A|$ of a $(p+1)$–linear map A is equal to p.

One extends the composition of linear operators to multilinear operators; if $A_1\in\mathcal{C}^{m_1}, A_2\in\mathcal{C}^{m_2}$, then:

$$(A_1\circ A_2)(f_1,...,f_{m_1+m_2+1}) :=$$
$$\sum_{j=1}^{m_1}(-1)^{(m_2)(j-1)}A_1(f_1,...,f_{j-1},A_2(f_j,...,f_{j+m_2}),f_{j+m_2+1},\ldots,f_{m_1+m_2+1})$$

for any (m_1+m_2+1)- tuple of elements of A. The **Gerstenhaber bracket** is defined by

$$[A_1,A_2]_G := A_1\circ A_2 - (-1)^{m_1m_2}A_2\circ A_1.$$

The differential d_D is defined by

$$d_D A = -[\mu, A] = -\mu\circ A + (-1)^{|A|}A\circ\mu$$

where μ is the usual product in the algebra A.

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Proposition 4.1. *The graded Lie algebra $\mathcal{C}$ together with the differential d_D is a differential graded Lie algebra.*

Here we consider the algebra $A = C^\infty(M)$, and we deal with the subalgebra of $\mathcal{C}$ consisting of multidifferential operators $\mathcal{D}_{poly}(M) := \bigoplus \mathcal{D}^i_{poly}(M)$ with $\mathcal{D}^i_{poly}(M)$ consisting of multi differential operators acting on $i+1$ smooth functions on M and vanishing on constants. It is easy to check that $\mathcal{D}_{poly}(M)$ is closed under the Gerstenhaber bracket and under the differential d_D, so that it is a DGLA.

Proposition 4.2. *An element $C \in \nu\mathcal{D}^1_{poly}(M)[[\nu]]$ (i.e. a series of bidifferential operator on the manifold M) yields a deformation of the usual associative pointwise product of functions μ:*

$$* = \mu + C$$

which defines a differential star product on M if and only if

$$d_D C - \frac{1}{2}[C, C]_G = 0.$$

Formal Poisson structures and the DGLA of multivector fields

A k**-multivector field** is a section of the k-th exterior power $\Lambda^k TM$ of the tangent space TM; the bracket of multivectorfields is the **Schouten-Nijenhuis bracket** which extends the usual Lie bracket of vector fields

$$\begin{aligned}&[X_1 \wedge \ldots \wedge X_k, Y_1 \wedge \ldots \wedge Y_l]_S \\ &\quad = \sum_{r=1}^{k}\sum_{s=1}^{l}(-1)^{r+s}[X_r, X_s]X_1 \wedge \ldots \widehat{X_r} \wedge \ldots \wedge X_k \wedge Y_1 \wedge \ldots \widehat{Y_s} \wedge \ldots \wedge Y_l.\end{aligned}$$

Since the bracket of an r- and an s-multivector fields on M is an $r+s-1$- multivector field, we define a structure of graded Lie algebra on the space $\mathcal{T}_{poly}(M)$ of multivector fields on M by setting $\mathcal{T}^i_{poly}(M)$ the set of skewsymmetric contravariant $i+1$-tensorfields on M (observe again a shift in the grading). We shall consider here

$$[T_1, T_2]'_S := -[T_2, T_1]_S.$$

Then $\mathcal{T}_{poly}(M)$ is turned into a differential graded Lie algebra setting the differential d_T to be identically zero.

Proposition 4.3. *An element $P \in \nu\mathcal{T}^1_{poly}(M)[[\nu]]$ (i.e. a series of bivectorfields on the manifold M) defines a formal Poisson structure on M if and only if*

$$d_T P - \frac{1}{2}[P, P]'_S = 0.$$

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L_∞-algebras and L_∞-morphisms

If one could construct an isomorphism of DGLA (i.e. a linear bijection which commute with the differentials and the brackets) between the algebra $\mathcal{T}_{poly}(M)$ of multivector fields and the algebra $\mathcal{D}_{poly}(M)$ of multidifferential operators, this would give a correspondence between a formal Poisson tensor on M and a formal differential star product on M. The natural map

$$U_1 \colon \mathcal{T}^i_{poly}(M) \longrightarrow \mathcal{D}^i_{poly}(M)$$

which extends the usual identification between vector fields and first order differential operators, is defined by:

$$U_1(X_0 \wedge \ldots \wedge X_n)(f_0, \ldots, f_n) = \frac{1}{(n+1)!} \sum_{\sigma \in S_{n+1}} \epsilon(\sigma)\, X_0(f_{\sigma(0)}) \cdots X_n(f_{\sigma(n)}).$$

Unfortunately this map fails to preserve the Lie structure.

One extends the notion of morphism between two DGLA to construct a morphism whose first order approximation is this map U_1. To do this one introduces the notion of L_∞-morphism.

A toy picture of our situation (finding a correspondence between a formal Poisson tensor P on M and a formal differential star product $* = \mu + C$ on M) is the following. If C and P were elements in neighborhoods of zero V_1 and V_2 of finite dimensional vector spaces, one could consider analytic vector fields X_1 on V_1, X_2 on V_2, vanishing at zero, given by $(X_1)_C = d_D C - \frac{1}{2}[C,C]_G$, $(X_2)_P = d_T P - \frac{1}{2}[P,P]'_S$ and one would be interested in finding a correspondence between zeros of X_2 and zeros of X_1. An idea would be to construct an analytic map $\phi : V_2 \to V_1$ so that $\phi(0) = 0$ and $\phi_* X_2 = X_1$. Such a map can be viewed as an algebra morphism $\phi^* : A_1 \to A_2$ where A_i is the algebra of analytic functions on V_i vanishing at zero. The vector field X_i can be seen as a derivation of the algebra A_i. A real analytic function being determined by its Taylor expansion at zero, one can look at $C(V_i) := \sum_{n \geq 1} S^n(V_i)$ as the dual space to A_i; it is a coalgebra. One view the derivation of A_i corresponding to the vector field X_i dually as a coderivation Q_i of $C(V_i)$. One is then looking for a coalgebra morphism $F : C(V_2) \to C(V_1)$ so that $F \circ Q_2 = Q_1 \circ F$.

This is generalized to the framework of graded algebras with the notion of L_∞-morphism between L_∞-algebras.

Definition 4.3. A **graded coalgebra** on the base ring $\mathbb{K}$ is a $\mathbb{Z}$–graded vector space $C = \bigoplus_{i \in \mathbb{Z}} C^i$ with a comultiplication, i.e. a graded linear map

$$\Delta \colon C \to C \otimes C$$

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such that $\Delta(C^i) \subset \bigoplus_{j+k=i} C^j \otimes C^k$ and such that (coassociativity):

$$(\Delta \otimes \mathrm{id})\Delta(x) = (\mathrm{id} \otimes \Delta)\Delta(x)$$

for every $x \in C$.

Additional structures that can be put on an algebra can be dualized to give a dual version on coalgebras.

Definition 4.4 (The coalgebra $C(V)$). *Let V is a graded vector space over $\mathbb{K}$, $V = \bigoplus_{i\in\mathbb{Z}} V^i$ and let $|v|$ denote the degree of $v \in V$. The tensor algebra is $T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}$ with $V^{\otimes 0} = \mathbb{K}$. It has the two quotients:*
the symmetric algebra $S(V) = T(V)/ < x \otimes y - (-1)^{|x||y|} y \otimes x >$, and the exterior algebra $\Lambda(V) = T(V)/ < x \otimes y + (-1)^{|x||y|} y \otimes x >$;
these spaces are naturally graded associative algebras. They can be given a structure of coalgebras with comultiplication Δ defined on a homogeneous element $v \in V$ by

$$\Delta v := 1 \otimes v + v \otimes 1$$

and extended as algebra homomorphism.
The ***reduced symmetric space*** *is $C(V) := S^+(V) := \bigoplus_{n>0} S^n(V)$.*

Definition 4.5. A **coderivation** of degree d on a graded coalgebra C is a graded linear map $\delta\colon C^i \to C^{i+d}$ which satisfies the (co–)Leibniz identity:

$$\Delta\delta(v) = \delta v' \otimes v" + (-1)^{d|v'|} v' \otimes \delta v"$$

if $\Delta v = \sum v' \otimes v"$. This can be rewritten with the usual Koszul sign conventions $\Delta\delta = (\delta \otimes \mathrm{id} + \mathrm{id} \otimes \delta)\Delta$.

Definition 4.6. A L_∞**–algebra** is a graded vector space V over $\mathbb{K}$ and a degree 1 coderivation Q defined on the reduced symmetric space $C(V[1])$ so that

$$Q \circ Q = 0. \tag{15}$$

[Given any graded vector space V, a new graded vector space $V[k]$ is defined by shifting the grading of the elements of V by k, i.e. $V[k] = \bigoplus_{i\in\mathbb{Z}} V[k]^i$ where $V[k]^i := V^{i+k}$.]

Definition 4.7. A L_∞**–morphism** between two L_∞–algebras, $F\colon (V,Q) \to (V',Q')$, is a morphism

$$F\colon C(V[1]) \longrightarrow C(V'[1])$$

of graded coalgebras, so that $F \circ Q = Q' \circ F$.

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In the same way that any algebra morphism from $S^+(V)$ to $S^+(V')$ is determined by its restriction to V and any derivation of $S^+(V)$ is determined by its restriction to V, a coalgebra–morphism F from the coalgebra $C(V)$ to the coalgebra $C(V')$ is uniquely determined by the composition of F and the projection $\pi' : C(V') \to V'$ and any coderivation Q of $C(V)$ is determined by the composition $F \circ \pi$ where π is the projection of $C(V)$ on V.

Definition 4.8. We call **Taylor coefficients of a coalgebra-morphism** $F\colon C(V) \to C(V')$ the sequence of maps $F_n\colon S^n(V) \to V'$ and **Taylor coefficients of a coderivation** Q of $C(V)$ the sequence of maps $Q_n\colon S^n(V) \to V$.

Given V and V' two graded vector spaces, any sequence of linear maps $F_n\colon S^n(V) \to V'$ of degree zero determines a unique coalgebra morphism $F\colon C(V) \to C(V')$ for which the F_n are the Taylor coefficients. Explicitely

$$F(x_1 \ldots x_n) = \sum_{j\geq 1} \frac{1}{j!} \sum_{\{1,\ldots,n\}=I_1\sqcup\ldots\sqcup I_j} \epsilon_x(I_1,\ldots,I_j) F_{|I_1|}(x_{I_1}) \cdots F_{|I_j|}(x_{I_j})$$

where the sum is taken over $I_1 \ldots I_j$ partition of $\{1,\ldots,n\}$ and $\epsilon_x(I_1,\ldots,I_j)$ is the signature of the effect on the odd x_i's of the unshuffle associated to the partition $(I_1,\ldots,I_j)$ of $\{1,\ldots,n\}$.

Similarly, if V is a graded vector space, any sequence $Q_n\colon S^n(V) \to V, n \geq 1$ of linear maps of degree i determines a unique coderivation Q of $C(V)$ of degree i whose Taylor coefficients are the Q_n. Explicitly

$$Q(x_1 \ldots x_n) = \sum_{\{1,\ldots,n\}=I\sqcup J} \epsilon_x(I,J)(Q_{|I|}(x_I)x_J.$$

The first conditions on the Taylor coefficients Q_n to have $Q^2 = 0$ are:

- $Q_1^2 = 0$ and Q_1 is a linear map of degree 1 on V;
- $Q_2(Q_1x.y + (-1)^{|x|-1}x.Q_1y) + Q_1Q_2(x.y) = 0$;
- $0 = Q_3(Q_1x.y.z + (-1)^{|x|-1}x.Q_1y.z + (-1)^{|x|+|y|-2}x.y.Q_1z) + Q_1Q_3(x.y.z) + Q_2(Q_2(x.y).z) + (-1)^{(|y|-1)(|z|-1)}Q_2(x.z).y + (-1)^{(|x|-1)(|y|+|z|-2)}Q_2(y.z).x$.

Defining

$$dx = (-1)^{|x|}Q_1x \qquad [x,y] := \overline{Q_2}(x \wedge y) = (-1)^{|x|(|y|-1)}Q_2(x,y), \quad (16)$$

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http://www.worldscibooks.com/mathematics/7352.html

the above relations show that d is a differential on V, and $[\ ,\ ]$ is a graded skewsymmetric bilinear map from $V \times V \to V$ satisfying

$$(-1)^{|x||z|}[[x,y],z] + (-1)^{|y||x|}[[y,z],x](-1)^{|z||y|}[[z,x],y] + \text{terms in } Q_3 = 0$$

and $d[x,y] = [dx,y] + (-1)^{|x|}[x,dy]$. In particular, we get:

Lemma 4.1. *Any L_∞–algebra (V,Q) so that all the Taylor coefficients Q_n of Q vanish for $n > 2$ yields a differential graded Lie algebra and vice versa.*

The first conditions for a sequence of linear maps $F_n : S^n(V[1]) \to V'[1]$ to be the Taylor coefficients of a L_∞–morphism between two L_∞–algebras (V,Q) and (V',Q'), i.e. so that $F \circ Q = Q' \circ F$ are

- $F_1 \circ Q_1 = Q'_1 \circ F_1$ so $F_1 : V \to V'$ is a morphism of complexes from (V,d) to (V',d')
- $F_1([x,y]) - [F_1x, F_1y]' =$ expression involving F_2

so there exist L_∞–morphisms between two DGLA's which are not DGLA–morphisms.

The equations for F to be a L_∞–morphism between two DGLA's (V,Q) and $(V',Q'$ (with $Q_n = 0, Q'_n = 0\ \forall n > 2)$ are

$$Q'_1 F_n(x_1 \cdot \ldots \cdot x_n) + \frac{1}{2} \sum_{\substack{U \sqcup J = \{1, ldots, n\} \\ I,J \neq \emptyset}} \epsilon_x(I,J) Q'_2(F_{|I|}(x_I) \cdot F_{|J|}(x_J))$$

$$= \sum_{k=1}^{n} \epsilon_x(k,1,\ldots\hat{k}\ldots,n) F_n(Q_1(x_k) \cdot x_1 \cdot \ldots \hat{x_k} \ldots \cdot x_n) \qquad (*)$$

$$+ \frac{1}{2} \sum_{k \neq l} \epsilon_x(k,l,1,\ldots\hat{kl}\ldots,n) F_{n-1}(Q_2(x_k \cdot x_l) \cdot x_1 \cdot \ldots \hat{x_k}\hat{x_l} \ldots \cdot x_n).$$

Formality, formal Poisson structures, and star products

Definition 4.9. Let $\mathfrak{m} = \nu\mathbb{R}[[\nu]]$. A $\mathfrak{m}$**– point** in a L_∞ algebra (V,Q) (over a field of characteristic zero) is an element $p \in \nu C(V)[[\nu]]$ so that $\Delta p = p \otimes p$; equivalently, it is an element $p = e^v - 1 = v + \frac{v^2}{2} + \cdots$ where v is an even element in $V[1] \otimes \mathfrak{m} = \nu V[1][[\nu]]$.

A solution of the generalized Maurer-Cartan equation is a $\mathfrak{m}$–point p where Q vanishes; equivalently, it is an odd element $v \in \nu V[[\nu]]$ so that

$$Q_1(v) + \frac{1}{2} Q_2(v \cdot v) + \cdots = 0. \qquad (17)$$

If $\mathfrak{g}$ is a DGLA, it is thus an element $v \in \mathfrak{g}$ so that $dv - \frac{1}{2}[v,v] = 0$.

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The image under a L_∞ morphism of a solution of the generalised Maurer-Cartan equation is again such a solution. In particular, if one builds a L_∞ morphism F betwwen the two DGLA we consider, $F : \mathcal{T}_{poly}(M) \to \mathcal{D}_{poly}(M)$, the image under F of the point $e^\alpha - 1$ corresponding to a formal Poisson tensor,

$$\alpha \in \nu \mathcal{T}^1_{poly}(M)[[\nu]] \text{ so that } [\alpha, \alpha]_S = 0, \tag{18}$$

yields a star product on M,

$$* = \mu + \sum_n F_n(\alpha^n). \tag{19}$$

Definition 4.10. Two L_∞–algebras (V, Q) and (V', Q') are **quasi-isomorphic** if there is a L_∞–morphism F so that $F_1 : V \to V'$ induces an isomorphism in cohomology.

Kontsevich has proven that if F is a L_∞–morphism between two L_∞–algebras (V, Q) and (V', Q') so that $F_1 : V \to V'$ induces an isomorphism in cohomology, then there exists a L_∞–morphism G between (V', Q') and (V, Q) so that $G_1 : V' \to V$ is a quasi inverse for F_1 .

Definition 4.11. Kontsevich's **formality** is a quasi isomorphism between the (L_∞–algebra structure associated to the) DGLA of multidifferential operators, $\mathcal{D}_{poly}(M)$, and its cohomology, which is the DGLA of multivector fields $\mathcal{T}_{poly}(M)$.

Such a formality induces a bijective correspondence between equivalence classes of formal Poisson structures and equivalence classes of star products.

4.2. *Kontsevich's formality for* $\mathbb{R}^d$

Kontsevich gave an explicit formula for the Taylor coefficients of a formality for $\mathbb{R}^d$, i.e. the Taylor coefficients F_n of an L_∞–morphism between the two DGLA's

$$F : (\mathcal{T}_{poly}(\mathbb{R}^d), Q) \to (\mathcal{D}_{poly}(\mathbb{R}^d), Q')$$

where Q corresponds to the DGLA of $(\mathcal{T}_{poly}(\mathbb{R}^d)\ ,\ [\ ,\ ]'_S\ ,\ D_T = 0)$ and Q' corresponds to the DGLA $(\mathcal{D}_{poly}(\mathbb{R}^d)\ ,\ [\ ,\ ]_G\ ,\ d_D)$, with the first coefficient

$$F_1 = U_1 : \mathcal{T}_{poly}(\mathbb{R}^d) \to \mathcal{D}_{poly}(\mathbb{R}^d)$$

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$U_1(X_0 \wedge \ldots \wedge X_n)(f_0, \ldots, f_n) = \frac{1}{(n+1)!} \sum_{\sigma \in S_{n+1}} \epsilon(\sigma) \quad X_0(f_{\sigma(0)}) \cdots X_n(f_{\sigma(n)})$. The formula writes

$$F_n = \sum_{m \geq 0} \sum_{\vec{\Gamma} \in G_{n,m}} \mathcal{W}_{\vec{\Gamma}} B_{\vec{\Gamma}}$$

- where $G_{n,m}$ is a set of oriented admissible graphs;
  [An admissible graph $\vec{\Gamma} \in G_{n,m}$ has n aerial vertices labelled $p_1, \ldots, p_n$, and m ground vertices labelled $q_1, \ldots, q_m$. From each aerial vertex p_i, a numer k_i of arrows are issued; each of them can end on any vertex except p_i but there can not be multiple arrows. There are no arrows issued from the ground vertices. One gives an order to the vertices:$(p_1, \ldots, p_n, q_1, \ldots, q_m)$ and one gives a compatible order to the arrows, labeling those issued from p_i with $(k_1 + \ldots + k_{i-1} + 1, \ldots, k_1 + \ldots + k_{i-1} + k_i)$. The arrows issued from p_i are named $\text{Star}(p_i) = \{\overrightarrow{p_i a_1}, \ldots, \overrightarrow{p_i a_{k_i}}\}$ with $\overrightarrow{v_{k_1+\ldots+k_{i-1}+j}} = \overrightarrow{p_i a_j}$.]
- where $B_{\vec{\Gamma}}$ associates a m–differential operator to an n–tuple of multi-vectorfields;
  [Given a graph $\vec{\Gamma} \in G_{n,m}$ and given n multivectorfields $(\alpha_1, \ldots, \alpha_n)$ on $\mathbb{R}^d$, one defines a m– differential operator $B_{\vec{\Gamma}}(\alpha_1 \cdot \ldots \cdot \alpha_n)$; it vanishes unless α_1 is a k_1–tensor, α_2 is a k_2–tensor,..., α_n is a k_n–tensor and in that case it is given by:

$$B_{\vec{\Gamma}}(\alpha_1 \cdot \ldots \cdot \alpha_n)(f_1, \ldots, f_m) = \sum_{i_1,..,i_K} D_{p_1} \alpha_1^{i_1 \ldots i_{k_1}} D_{p_2} \alpha_2^{i_{k_1+1} \ldots i_{k_1+k_2}} \ldots D_{p_n} \alpha_n^{i_{k_1+..+k_{n-1}+1} \ldots i_K} D_{q_1} f_1 \ldots D_{q_m} f_m$$

  where $K := k_1 + \cdots + k_n$ and where $D_a := \Pi_{j | \overrightarrow{v_j} = \overrightarrow{\cdot a}} \partial_{i_j}$.]
- where $\mathcal{W}_{\vec{\Gamma}}$ is the integral of a form $\omega_{\vec{\Gamma}}$ over the compactification of a configuration space $C^+_{\{p_1,\ldots,p_n\}\{q_1,\ldots,q_m\}}$.
  [Consider the upper half plane $\mathcal{H} = \{z \in \mathbb{C} | Im(z) > 0\}$; define

$$Conf^+_{\{z_1,\ldots,z_n\}\{t_1,\ldots,t_m\}} := \left\{z_1, \ldots, z_n, t_1, \ldots, t_m \,\middle|\, \begin{array}{l} z_j \in \mathcal{H}; z_i \neq z_j \text{ for } i \neq j; \\ t_j \in \mathbb{R}; t_1 < t_2 \cdots < t_m \end{array}\right\}$$

  and $C^+_{\{p_1,\ldots,p_n\}\{q_1,\ldots,q_m\}}$ to be the quotient of this space by the action of the 2-dimensional group G of all transformations of the form

$$z_j \mapsto a z_j + b \qquad t_i \mapsto a t_i + b \qquad a > 0, b \in \mathbb{R}.$$

  The configuration space $C^+_{\{p_1,\ldots,p_n\}\{q_1,\ldots,q_m\}}$ has dimension $2n + m - 2$ and has an orientation induced on the quotient by

$$\Omega_{\{z_1,\ldots,z_n;t_1,\ldots,t_m\}} = dx_1 \wedge dy_1 \wedge \ldots dx_n \wedge dy_n \wedge dt_1 \wedge \ldots \wedge dt_m$$

  if $z_j = x_j + i y_j$.

The compactification $\overline{C^+_{\{p_1,\ldots,p_n\}\{q_1,\ldots,q_m\}}}$ is defined as the closure of the image of the configuration space $C^+_{\{p_1,\ldots,p_n\}\{q_1,\ldots,q_m\}}$ into the product of a torus and the product of real projective spaces $P^2(\mathbb{R})$ under the map Ψ induced from a map ψ defined on $Conf^+_{\{z_1,\ldots,z_n\}\{t_1,\ldots,t_m\}}$ in the following way: to any pair of distinct points A, B taken amongst the $\{z_j, \overline{z}_j, t_k\}$ ψ associates the angle $\arg(B-A)$ and to any triple of distinct points A, B, C in that set, ψ associates the element of $P^2(\mathbb{R})$ which is the equivalence class of the triple of real numbers $(|A-B|, |B-C|, |C-A|)$.
Given a graph $\vec{\Gamma} \in G_{n,m}$, one defines a form on $\overline{C^+_{\{p_1,\ldots,p_n\}\{q_1,\ldots,q_m\}}}$ induced by

$$\omega_{\vec{\Gamma}} = \frac{1}{(2\pi)^{k_1+\ldots+k_n}(k_1)!\ldots(k_n)!} d\Phi_{\overrightarrow{v_1}} \wedge \ldots \wedge d\Phi_{\overrightarrow{v_K}}$$

where $\Phi_{\overrightarrow{p_j a}}$ =Arg$(\frac{a-p_j}{a-\overline{p}_j})$.]

We give here a sketch of the proof; a detailed proof can be found in [1].

Remark that $\mathcal{W}_{\vec{\Gamma}} \neq 0$ implies that the dimension of the configuration space $2n+m-2$ is equal to the degree of the form $= k_1+\ldots+k_n = K$(=the number of arrows in the graph).

We shall write

$$F_n = \sum_{m\geq 0} \sum_{\vec{\Gamma}\in G_{n,m}} \mathcal{W}_{\vec{\Gamma}} B_{\vec{\Gamma}} = \sum F_{(k_1,\ldots,k_n)}$$

where $F_{(k_1,\ldots,k_n)}$ corresponds to the graphs $\vec{\Gamma} \in G_{n,m}$ with k_i arrows starting from p_i. The formality equation reads:

$$\begin{aligned} 0 = F_{(k_1,\ldots,k_n)}(\alpha_1 \cdot \quad \cdot \alpha_n) \circ \mu - (-1)^{\sum k_i - 1} \mu \circ F_{(k_1,\ldots,k_n)}(\alpha_1 \cdot \quad \cdot \alpha_n) \\ + \sum_{\substack{U \sqcup J = \{1, ldots, n\} \\ I,J\neq\emptyset}} \epsilon_\alpha(I,J)(-1)^{(|k_I|-1)|k_J|} F_{(k_I)}(\alpha_I) \circ F_{(k_J)}(\alpha_J) \\ - \sum_{i\neq j} F\epsilon_x(i,j,1,..\hat{i}\hat{j}..,n) F_{(k_i+k_j-1,k_1,.\hat{k_i}\hat{k_j}.,k_n)}((\alpha_i \bullet \alpha_j) \cdot \alpha_1 \cdot ..\hat{\alpha_i}\hat{\alpha_j}.. \cdot \alpha_n) \end{aligned}$$

where

$$\alpha_1 \bullet \alpha_2 = \frac{k_1}{(k_1)!(k_2)!} \alpha_1^{ri_1\ldots i_{k_1-1}} \partial_r \alpha_2^{j_1\ldots j_{k_2}} \partial_{i_1} \wedge \cdots \wedge \partial_{i_{k_1-1}} \wedge \partial_{j_1} \wedge \cdots \wedge \partial_{j_{k_2}}$$

so that $[\alpha_1, \alpha_2]_S = (-1)^{k_1-1} \alpha_1 \bullet \alpha_2 - (-1)^{k_1(k_2-1)} \alpha_2 \bullet \alpha_1$. The right hand side of the formality equation can be written as

$$\sum_{\overrightarrow{\Gamma'}} C_{\overrightarrow{\Gamma'}} B_{\overrightarrow{\Gamma'}}(\alpha_1 \cdot \quad \cdot \alpha_n)$$

for graphs $\overrightarrow{\Gamma'}$ with n aerial vertices, m ground vertices and $2n+m-3$ arrows.

To a face G of codimension 1 in the boundary of $\overline{C^+_{\{p_1,\ldots,p_n\}\{q_1,\ldots,q_m\}}}$ and an oriented graph $\overrightarrow{\Gamma'}$ as above, one associates one term in the formality equation (or 0).

• $G = \partial_{\{p_{i_1},..,p_{i_{n_1}}\}\{q_{l+1},..,q_{l+m_1}\}} C^+_{\{p_1,..s,p_n\}\{q_1,..,q_m\}}$ if the aerial points $\{p_{i_1},..,p_{i_{n_1}}\}$ and the ground points $\{q_{l+1},..,q_{l+m_1}\}$ all collapse into a ground point q. We associate to G the operator $B'_{\overrightarrow{\Gamma'},G}(\alpha_1 \cdot \quad \cdot \alpha_n)$ which is the term in the formality equation of the form $B_{\overrightarrow{\Gamma'}}$ obtained from

$$B_{\overrightarrow{\Gamma_2}}(\alpha_{j_1} \cdot \quad \cdot \alpha_{j_{n_2}})(f_1,..,f_l, B_{\overrightarrow{\Gamma_1}}(\alpha_{i_1} \cdot \quad \cdot \alpha_{i_{n_1}})(f_{l+1},..,f_{l+m_1}), f_{l+m_1+1},..,f_m)$$

where $\overrightarrow{\Gamma_1}$ is the restriction of $\overrightarrow{\Gamma'}$ to $\{p_{i_1},..,p_{i_{n_1}}\} \cup \{q_{l+1},..,q_{l+m_1}\}$, where $\overrightarrow{\Gamma_2}$ is obtained from $\overrightarrow{\Gamma'}$ by collapsing $\{p_{i_1},\ldots,p_{i_{n_1}}\} \cup \{q_{l+1},\ldots,q_{l+m_1}\}$ into q and where $\{j_1 < \ldots < j_{n_2}\} = \{1,\ldots,n\} \setminus \{i_1,\ldots,i_{n_1}\}$.

• $G = \partial_{\{p_i,p_j\}} C^+_{\{p_1,\ldots,p_n\}\{q_1,\ldots,q_m\}}$ if the aerial points $\{p_i,p_j\}$ collapse into an aerial point p. If the arrow $\overrightarrow{p_ip_j}$ belongs to $\overrightarrow{\Gamma'}$, we associate to G the operator $B'_{\overrightarrow{\Gamma'},G}(\alpha_1 \cdot \quad \cdot \alpha_n)$ which is the term in the formality equation of the form $B_{\overrightarrow{\Gamma'}}$ obtained from

$$B_{\overrightarrow{\Gamma_2}}(\alpha_i \bullet \alpha_j) \cdot \alpha_1 \cdot \quad \hat{\alpha}_i \hat{\alpha}_j \quad \cdot \alpha_n)$$

where $\overrightarrow{\Gamma_2}$ is obtained from $\overrightarrow{\Gamma'}$ by collapsing $\{p_i,p_j\}$ into p, discarding the arrow $\overrightarrow{p_ip_j}$.

If $\overrightarrow{p_ip_j}$ is not an arrow in $\overrightarrow{\Gamma'}$, we set $B'_{\overrightarrow{\Gamma'},G}(\alpha_1 \cdot \quad \cdot \alpha_n) = 0$.

• $G = \partial_{\{p_{i_1},\ldots,p_{i_{n_1}}\}} C^+_{\{p_1,\ldots,p_n\}\{q_1,\ldots,q_m\}}$ if the aerial points $\{p_{i_1},\ldots,p_{i_{n_1}}\}$ all collapse with $n_1 > 2$. We associate to such a face G, the operator $B'_{\overrightarrow{\Gamma'},G} = 0$.

Looking at the coefficients of $B_{\overrightarrow{\Gamma'}}$ in each of the $B'_{\overrightarrow{\Gamma'},G}$, the right hand side of the formality equation now writes

$$\sum_{\overrightarrow{\Gamma'}} C_{\overrightarrow{\Gamma'}} B_{\overrightarrow{\Gamma'}}(\alpha_1 \cdot \quad \cdot \alpha_n) = \sum_{\overrightarrow{\Gamma'}} \sum_{G \subset \partial C^+} B'_{\overrightarrow{\Gamma'},G}(\alpha_1 \cdot \quad \cdot \alpha_n)$$

$$= \sum_{\overrightarrow{\Gamma'} \in G_{n,m}} \Big(\sum_{G \subset \partial C^+} \int_G \omega_{\overrightarrow{\Gamma'}} \Big) B_{\overrightarrow{\Gamma'}}(\alpha_1 \cdot \quad \cdot \alpha_n) = 0$$

by Stokes theorem on the manifold with corners which is the compactification of C^+.

Observe that the explicit formula for the Taylor's coefficients F_j of Kontsevich L_∞ morphism from the differential graded Lie algebra of polyvectorfields on $\mathbb{R}^d$ to the differential graded Lie algebra of polydifferential operators on $\mathbb{R}^d$ shows that the coefficients of the multidifferential operator $F_j(\alpha_1,\ldots,\alpha_j)$ are given by multilinear universal expressions in the partial derivatives of the coefficients of the multivectorfields $\alpha_1,\ldots,\alpha_j$.
Hence the corresponding star product

$$f *_K^P g = f\,g + \sum_{n=1}^{\infty} \frac{\nu^n}{n!} F_n(P,\ldots,P)(f,\,g) = fg + \nu P(df,dg) + O(\nu^2) \quad (20)$$

is natural and defined by bidifferential operator whose coefficients are universal polynomials of degree n in the partial derivatives of the coefficients of the tensor P.

4.3. *Universal star product and universal formality*

Kontsevich also obtained the existence of star products on a general Poisson manifold using abstract arguments. A more direct construction of a star product on a d-dimensional Poisson manifold (M,P), using Kontsevich's formality on $\mathbb{R}^d$, was given by Cattaneo, Felder and Tomassini in [6]. Using a linear torsionfree connection ∇ on the manifold M, the construction starts with the identification of the commutative algebra $C^\infty(M)$ of smooth functions on M with the algebra of flat sections of the jet bundle $E \to M$, for the Grothendieck connection D^G. Let us recall this construction.

The jet bundle and Grothendieck connection

Let M be a d-dimensional manifold and consider the **jet bundle** $E \to M$ (the bundle of infinite jet of functions) with fibers $\mathbb{R}[[y^1,\ldots,y^d]]$ (i.e. formal power series in $y \in \mathbb{R}^d$ with real coefficients) and transition functions induced from the transition functions of the tangent bundle TM:

$$E = F(M) \times_{Gl(d,\mathbb{R})} \mathbb{R}[[y^1,\ldots,y^d]] \quad (21)$$

where $F(M)$ is the frame bundle. A section $s \in \Gamma(E)$ can be written in the form

$$s = s(x;y) = \sum_{p=0}^{\infty} s_{i_1\ldots i_p}(x) y^{i_1} \cdots y^{i_p}$$

where the $s_{i_1\ldots i_p}$ are components of symmetric covariant tensors on M.

The exponential map for the connection ∇ gives an identification

$$\exp_x : U \cap T_xM \to M \quad y \mapsto \exp_x(y) \tag{22}$$

at each point x, of the intersection of the tangent space T_xM with a neighborhood U of the zero section of the tangent bundle TM with a neighborhood of x in M.

To a function $f \in C^\infty(M)$, one associates the section f_ϕ of the jet bundle $E \to M$ given, for any $x \in M$, by the Taylor expansion at $0 \in T_xM$ of the pullback $f \circ \exp_x$; it is given by:

$$f_\phi(x; y) = f(x) + \sum_{n>0} \frac{1}{n!} \nabla^n_{i_1 \dots i_n} f(x)\, y^{i_1} \dots y^{i_n}. \tag{23}$$

The **Grothendieck connection** D^G on E is defined by:

$$D^G_X s(x; y) := \frac{d}{dt}_{|t=0} s(x(t); \exp^{-1}_{x(t)}(\exp_x(y))) \tag{24}$$

for any curve $t \to x(t) \in M$ representing $X \in T_xM$ and for any $s \in \Gamma(E)$. From the definition, it is clear that D^G is flat and that $D_G f_\phi = 0$ for any $f \in C^\infty(M)$.

Introducing $\delta = \sum_i dx^i \frac{\partial}{\partial y^i}$, and $\nabla' = \sum_i dx^i \left(\partial_{x^i} - \sum_{jk} \Gamma^k_{ij}\, y^j\, \partial_{y^k}\right)$ one can write

$$D^G = -\delta + \nabla' + A, \tag{25}$$

where A is a 1-form on M with values in the fiberwize vectorfields on E,

$$A(x; y) =: \sum_{ik} dx^i\, A^k_i(x; y)\, \partial_{y^k} = \sum_{ik} dx^i \left(-\frac{1}{3} \sum_{rs} R^k_{ris}(x) y^r y^s + 0(y^3)\right) \partial_{y^k}. \tag{26}$$

One extends D^G to the space $\Omega(M, E)$ of E-valued forms on M:

$$D^G = -\delta + \nabla' + A \ \text{ with } \nabla' = d - \sum_{ijk} dx^i \Gamma^k_{ij}\, y^j\, \partial_{y^k}. \tag{27}$$

Introducing (as in Fedosov"s construction) $\delta^* = \sum_j y^j\, i(\frac{\partial}{\partial x^j})$ on $\Omega(M, E)$, we have $(\delta^*)^2 = 0, \delta^2 = 0$ and for any $\omega \in \Omega^q(M, E_p)$, i.e. a q-form of degree p in y, we have $(\delta\delta^* + \delta^*\delta)\omega = (p + q)\omega$. Hence, defining, for any $\omega \in \Omega^q(M, E_p)$

$$\begin{aligned} \delta^{-1}\omega &= \frac{1}{p+q} \delta^* \omega && \text{when } p + q \neq 0 \\ &= 0 && \text{when } p = q = 0 \end{aligned}$$

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we see that any δ-closed q-form ω of degree p in y, when $p+q>0$, writes uniquely as $\omega=\delta\sigma$ with $\delta^*\sigma=0$; σ is given by $\sigma=\delta^{-1}\omega$.

One proceeds by induction on the degree in y to see that the cohomoly of D^G is concentrated in degree 0 and that any flat section of E is determined by its part of degree 0 in y. Remark that given any section s of E then $s(x;y=0)$ determines a smooth function f on M. If $D^Gs=0$, then $s-f_\phi$ is still D^G closed. By the above, Its terms of lowest order in y must be of the form $\delta\sigma$ hence must vanish since we have a 0-form. Hence we have:

Lemma 4.2. *[5] Any section of the jet bundle $s\in\Gamma(E)$ is the Taylor expansion of the pullback of a smooth function f on M via the exponential map of the connection ∇ if and only if it is horizontal for the Grothendieck-connection D^G:*

$$s=f_\phi \text{ for a } f\in C^\infty(M) \Leftrightarrow s\in\Gamma_{hor}(E):=\{\,s'\in\Gamma(E)\,|\,D^Gs'=0\,\}. \quad (28)$$

Furthermore, the cohomology of D^G is concentrated in degree 0.

Lemma 4.3. *[2] The 1-form A on M with values in the fiberwize vector-fields on E is given by $A(x;y)=:\sum_{ik}dx^i\,A_i^k(x;y)\,\partial_{y_k}$ where the A_i^k are universal polynomials given by concatenations of iterative covariant derivatives of the curvature. This 1-form A is uniquely characterized by the fact that $\delta^{-1}A=0$. and the fact that $D^G=-\delta+\nabla'+A$ is flat.*

Universal star product

The construction of a star product on any Poisson manifold by Cattaneo, Felder and Tomassini proceeds as follows: one quantize the identification of the commutative algebra of smooth fuctions on M with the algebra of flat sections of E in the following way.

A deformed algebra stucture on $\Gamma(E)[[\nu]]$ is obtained through fiberwize quantization of the jet bundle using Kontsevich star product on $\mathbb{R}^d$.

Consider the fiberwize Poisson structure on E, P_ϕ, which is given, at a point $x\in M$, by the Taylor expansion (infinite jet) at $y=0$ of the push-forward $(\exp_x)_*^{-1}P(\exp_x y)$.

One then considers fiberwize Kontsevich star product on $\Gamma(E)[[\nu]]$:

$$\sigma *_K^{P_\phi}\tau=\sigma\tau+\sum_{n=1}^{\infty}\frac{\nu^n}{n!}F_n(P_\phi,\dots,P_\phi)(\sigma,\tau).$$

The operator D_X^G is not a derivation of this deformed product; one constructs a flat covariant derivative of the sections of E, D, which is a deriva-

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tion of $*_K^{P_\phi}$. One defines first the derivation

$$D^1_X = X + \sum_{j=0}^{\infty} \frac{\nu^j}{j!} F_{j+1}(\hat{X}, P_\phi, \ldots, P_\phi) \tag{29}$$

where $\hat{X} := D^G_X - X$ is a vertical vectorfield on E. The connection D^1 is not flat so one deforms it by

$$D := D^1 + [\gamma, \cdot]_{*_K^{P_\phi}}$$

so that D is flat. The 1-form γ is constructed inductively using the fact that the cohomology of D^G vanishes.

The next point is to identify series of functions on M with the algebra of flat sections of this quantized bundle of algebras to define the star product on M.

This is done by buildind a map $\rho : \Gamma(E)[[\nu]] \to \Gamma(E)[[\nu]]$ so that $\rho \circ D^G = D \circ \rho$. This map is again constructed by induction using the vanishing of the cohomology.

It results [2] from the explicit expression of the form A and the operator δ^{-1} that the star product constructed in this way is universal.

Universal formality

Similarly, Dolgushev [8] constructs a L_∞ morphism from the differential graded Lie algebra of polyvectorfields on M to the differential graded Lie algebra of polydifferential operators on M.

He defines a resolution of polydifferential operators and polyvectorfields on M using the complexes $(\Omega(M, \mathcal{D}_{poly}), D_F^{\mathcal{D}_{poly}})$ and $(\Omega(M, \mathcal{T}_{poly}), D_F^{\mathcal{T}_{poly}})$ where $\mathcal{T}_{poly}$ is the bundle of formal fiberwize polyvectorfields on E and $\mathcal{D}_{poly}$ is the bundle of formal fiberwize polydifferential operators on E. A section of $\mathcal{T}^k_{poly}$ is of the form

$$\mathcal{F}(x;y) = \sum_{n=0}^{\infty} \mathcal{F}^{j_1 \ldots j_{k+1}}_{i_1 \ldots i_n}(x) y^{i_1} \ldots y^{i_n} \frac{\partial}{\partial y^{j_1}} \wedge \ldots \wedge \frac{\partial}{\partial y^{j_{k+1}}}, \tag{30}$$

where $\mathcal{F}^{j_1 \ldots j_{k+1}}_{i_1 \ldots i_n}(x)$ are coefficients of tensors, symmetric in the covariant indices $i_1, \ldots, i_n$ and antisymmetric in the contravariant indices $j_1, \ldots, j_{k+1}$. A section of $\mathcal{D}^k_{poly}$ is of the form

$$\mathcal{O}(x;y) = \sum_{n=0}^{\infty} \mathcal{O}^{\alpha_1 \ldots \alpha_{k+1}}_{i_1 \ldots i_n}(x) y^{i_1} \ldots y^{i_n} \frac{\partial^{|\alpha_1|}}{\partial y^{\alpha_1}} \otimes \ldots \otimes \frac{\partial^{|\alpha_{k+1}|}}{\partial y^{\alpha_{k+1}}}, \tag{31}$$

where the α_l are multi-indices and $\mathcal{O}^{\alpha_1\ldots\alpha_{k+1}}_{i_1\ldots i_n}(x)$ are coefficients of tensors symmetric in the covariant indices $i_1,\ldots,i_n$. and symmetric in each block of α_i contravariant indices.

The spaces $\Omega(M,\mathcal{T}_{poly})$ and $\Omega(M,\mathcal{D}_{poly})$ have a formal fiberwise DGLA structure: the degree of an element in $\Omega(M,\mathcal{T}_{poly})$ (resp. $\Omega(M,\mathcal{D}_{poly})$) is defined by the sum of the degree of the exterior form and the degree of the polyvector field (resp. the polydifferential operator), the bracket on $\Omega(M,\mathcal{T}_{poly})$ is defined by $[\omega_1\otimes\mathcal{F}_1,\omega_2\otimes\mathcal{F}_2]_{SN} := (-1)^{k_1q_2}\omega_1\wedge\omega_2\otimes[\mathcal{F}_1,\mathcal{F}_2]_{SN}$ for ω_i a q_i form and $\mathcal{F}_i$ a section in $\mathcal{T}^{k_i}_{poly}$ and similarly for $\Omega(M,\mathcal{D}_{poly})$ using the Gerstenhaber bracket. The differential on $\Omega(M,\mathcal{T}_{poly})$ is 0 and the differential on $\Omega(M,\mathcal{D}_{poly})$ is defined by $\partial := [m_{pf},.]_G$ where m_{pf} is the fiberwize multiplication of formal power series in y of E.

The differential $D^{\mathcal{T}_{poly}}_G$ is defined on $\Omega(M,\mathcal{T}_{poly})$ by

$$D^{\mathcal{T}_{poly}}_G\mathcal{F} := \nabla^{\mathcal{T}_{poly}}\mathcal{F} - \delta^{\mathcal{T}_{poly}}\mathcal{F} + [A,\mathcal{F}]_{SN} \tag{32}$$

with $\nabla^{\mathcal{T}_{poly}}\mathcal{F} = d\mathcal{F} - \left[\sum_{ijk} dx^i\Gamma^k_{ij}\,y^j\,\partial_{y_k},\ \mathcal{F}\right]_{SN}$, $\delta^{\mathcal{T}_{poly}}\mathcal{F} = \left[\sum_i dx^i\frac{\partial}{\partial y^i},\ \mathcal{F}\right]_{SN}$. Similarly $D^{\mathcal{D}_{poly}}_F$ is defined on $\Omega(M,\mathcal{D}_{poly})$ by

$$D^{\mathcal{D}_{poly}}_F\mathcal{O} := \nabla^{\mathcal{D}_{poly}}\mathcal{O} - \delta^{\mathcal{D}_{poly}}\mathcal{O} + [A,\mathcal{O}]_G \tag{33}$$

with $\nabla^{\mathcal{D}_{poly}}$ and $\delta^{\mathcal{D}_{poly}}$ defined as above with the Gerstenhaber bracket.

Again the cohomology is concentrated in degree 0 and a flat section $\mathcal{F}\in\mathcal{T}_{poly}$ or $\mathcal{O}\in\mathcal{D}_{poly}$ is determined by its terms $\mathcal{F}_0$ or $\mathcal{O}_0$ of order 0 in y.

We associate to a polyvector field $F\in T^k_{poly}(M)$ a section $F_\phi\in\Gamma(\mathcal{T}_{poly})$: for a point $x\in M$ one considers the Taylor expansion (infinite jet) $F_\phi(x;y)$ at $y=0$ of the push-forward $(\exp_x)^{-1}_*F(\exp_x y)$. Clearly this definition implies that $X_\phi(f_\phi) = (Xf)_\phi$ so that F_ϕ is uniquely determined by the fact that

$$F_\phi(f^1_\phi,\ldots,f^{k+1}_\phi) = \left(F(f^1,\ldots,f^{k+1})\right)_\phi \qquad \forall\, f^j\in C^\infty(M). \tag{34}$$

Similarly we associate to a differential operator $O\in D^k_{poly}(M)$ a section $O_\phi\in\Gamma(\mathcal{D}_{poly})$ determined by the fact that

$$O_\phi(f^1_\phi,\ldots,f^{k+1}_\phi) = \left(O(f^1,\ldots,f^{k+1})\right)_\phi \qquad \forall\, f^j\in C^\infty(M). \tag{35}$$

Observe that $D_G^{\mathcal{T}_{poly}} F_\phi = 0$ and similarly $D_G^{\mathcal{D}_{poly}} O_\phi = 0$, hence:

Lemma 4.4. *[2, 8] A section of $\mathcal{T}_{poly}$ is $D_F^{\mathcal{T}_{poly}}$–horizontal if and only if is a Taylor expansion of a polyvectorfield on M, i.e. iff it is of the form F_ϕ for some $F \in T^k_{poly}(M)$; a section of $\mathcal{D}_{poly}$ is $D_F^{\mathcal{D}_{poly}}$–horizontal if and only if is of the form O_ϕ for some $O \in D^k_{poly}(M)$.*

Dolgushev constructs his L_∞-morphism in two steps from the fiberwize Kontsevich formality from $\Omega(M, \mathcal{T}_{poly})$ to $\Omega(M, \mathcal{D}_{poly})$ building first a twist which depends only on the curvature and its covariant derivatives, then building a contraction using the vanishing of the D_G cohomology. Hence the Taylor coefficients of this L_∞-morphism, which are. a collection of maps F_j^D associating to j multivectorfields α_k on M a multidifferential operator $F_j^D(\alpha_1, \ldots, \alpha_j)$ are such that the tensors defining this operator are given by universal polynomials in the tensors defining the α_j's, the curvature tensor and their iterated cavariant derivatives.

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