Perspectives in Mathematical Sciences
Interdisciplinary Mathematical Sciences, Volume 9, 2009
pp. 1–36

Chapter 1

Periodic Boundary Problems for Analytic Function Including Automorphic Functions

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Two kinds of fundamental boundary problems for analytic functions, including automorphic functions, and their applications are discussed.

1. Periodic Boundary Value Problems for Analytic Functions

As a mathematical tool for the investigation of periodic problem in plane elasticity, two kinds of fundamental boundary value problems for analytic functions will be discussed in the present chapter, namely, the periodic Riemann boundary value problems, the periodic Riemann-Hilbert boundary value problems in the half plane. The more general formulations of these problems are those for automorphic functions which were studied by F. D. Gakhov and L.I. Chibrikova, Guoping Li and Youzhong Guo from the middle of the last century. However, the periodic problems discussed here are much more important in applications.

1.1. *Periodic Riemann Boundary Value Problems: Case of Closed Contours*

1) Formulation of the problems

Let L_k, $k = 0, \pm 1, \pm 2, \ldots$, be a set of smooth closed contours, non-intersecting to each other, oriented counter-clockwise, for the same shape and horizontally distributed with period $\pi a (a > 0)$.
The region interior to L_k is denoted by S_k^+, S_k^- is the complement of $S_k^+ + L_k$; and the region exterior to $L \equiv \sum_{k=-\infty}^{+\infty} L_k$ is denoted by S^-. We may always assume the origin $O \in S_0^+$ and all the points $\pm\frac{1}{2}\pi, \pm\frac{3}{2}\pi \ldots$ lie in S^-.
For briefness, we assume that L_k is a single closed smooth contour. In fact, the following discussions remain effective with slight modifications when L_k consists of a finite number of intersecting closed contours. Moreover, if L is an arbitrary smooth curve with period $a\pi$ (i.e., L_0 is a smooth curve from $-\frac{1}{2}a\pi + iy_0$ to $\frac{1}{2}a\pi + iy_0$ with

the same slope at its end-points), the following are also valid in case of suitable modifications.

2) Periodic Riemann boundary value problem (problem P_1): find a function $\Phi(z)$ in the complex plane with period $a\pi$, satisfying

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t),\ t \in L \tag{1.1}$$

where $\Phi^\pm(z)$ denote the boundary values (limiting value) of a sectionally holomorphic function $\Phi(z)$ (holomorphic in $S_k, k = 0, \pm1, \pm2, \cdots$, and S^-) from the positive (left) and the negative (right) sides of L respectively, while $G(t)$ and $g(t) \in H$ are given on L with period $a\pi$:

$$G(t + a\pi) = G(t), g(t + a\pi) = g(t), t \in L. \tag{1.2}$$

and $G(t) \neq 0$ (normal type).

Here, a functions $f(t) \in H(A, \mu)$ (function with Hölder Continuity) on L means

$$|f(t) - f(t')| \leq A|t - t'|^\mu, \quad \forall\, t, t' \in L$$

for certain positive constant A and $\mu(0 < \mu \leq 1)$. Note that ∞ is a limiting point of the points on L so that the solution $\Phi(z)$ of the problem (if any) could not have a definite limit as $z \to \infty$ in general. However, we may ask $\Phi(z)$ meeting certain requirements at $z = \pm\infty$ (that means , for $z = x + iy$, x may be arbitrary and $y \to \pm\infty$ respectively). We always require $\Phi(\pm\infty i)$ to be bounded (i.e. finite).

If $g(t) \equiv 0$, the problem is homogeneous, will be denoted by P_1^0, and otherwise, non-homogeneous.

Throughout this chapter, all periodic functions are always assumed with periodicity $\pi a(a > 0)$, for t an a definite limit.

Theorem 1 (Chibrikova) If $\Phi(\pm\infty i)$ are required to be bonded, then the homogeneous problem P_1^0 has $k + 1$ linearly independent solutions when its index $k \geq 0$ and has only the trivial solution when $k < 0$.

Theorem 2 (Chibrikova) If $\Phi(\pm\infty i)$ are required to be bonded, the general solution of the non- homogeneous problem P_1 has $k + 1$ arbitrary constants when $k \geq -1$; It is (uniquely) solvable when $k < -1$, iff $-k - 1$ conditions are valid as follows:

$$\int_{L_0} \frac{g(t)}{X^+(t)} \frac{\sin^{j-1} \frac{t}{a}}{\cos^{j+1} \frac{t}{a}} dt = 0, j = 1, \cdots, -k - 1;$$

In the case of index $k = 0$, then

$$\begin{aligned}\Gamma(z) &= \frac{1}{2a\pi i} \int_{L_0} \log K \cdot \cot \frac{t - z}{a} dt \\ &= \frac{\log K}{2\pi i} [\log \sin \frac{t - z}{a}]_{L_0} \\ &= \begin{cases} \log K, & when\, z \in S^+, \\ 0, & when\, z \in S^-. \end{cases}\end{aligned}$$

Where $\log K$ may be taken definitely and arbitrarily. Thus $X^+(z) = K,\ X^-(z) = 1$. Therefore, as $\Phi(\pm\infty i)$ are bounded, the general solution of the problem P_1 is

$$\Phi(z) = \begin{cases} \frac{1}{2a\pi i}\int_{L_0} g(t)\cot\frac{t-x}{a}dt + C, & z \in S^+, \\ \frac{1}{K}[\frac{1}{2a\pi i}\int_{L_0} g(t)\cot\frac{t-x}{a}dt + C], z \in S^-. \end{cases}$$

1.2. *Periodic Riemann Boundary Value Problems: Case of Open Ares or Discontinuous Coefficients*

1) Case of open arcs

Consider the case $L = \sum_{k=-\infty}^{+\infty} L_k$ being periodic as before when L_0 consists of p non-intersecting smooth open arcs $a_r b_r$, $r = 1, \cdots, p$, oriented from a_r to b_r and both $G(t), g(t) \in H(A, \mu)$ on each arc (including its end-points) with $G(t) \neq 0$. We get the general solution of problem P_1^0:

$$\Phi(z) = X(z)P_k(\tan\frac{z}{a}),$$

where the characteristic function

$$X(z) = \prod(z)e^{\Gamma(z)},$$

in which

$$\prod(z) = \prod_{i=1}^{2p}(\tan\frac{z}{a} - \tan\frac{c_j}{a})^{\lambda_i}$$

$$\Gamma(z) = \frac{1}{2a\pi i}\int_{L_0}\log G(t)\tan\frac{t-z}{a}dt.$$

When a constant factor is merged into, then

$$\Phi(z) = \prod_{i=1}^{2p}\sin^{\lambda_i}\frac{z-c_j}{a}e^{\Gamma(z)}Q_k(\sin\frac{z}{a}, \cos\frac{z}{a}).$$

2) An important special case

For the need of application later, consider the special case where L_0 consist of a single- line-segment $\gamma_0 : -l \le t \le l$ ($l < \frac{1}{2}a\pi$) along the real axis and $G(t) = -K$ is a negative real constant. We would seek for solutions of the corresponding problem P_1^0, i.e., solutions permitted to be integrably unbounded at not near $z = \pm l$. Now $c_1 = -l, c_2 = l$. The general solution is

$$\Phi(z) = \frac{1}{2a\pi i\sqrt{R(z)}}\int_{-l}^{l} g(t)\sqrt{R(t)}\cot\frac{t-z}{a}dt + \frac{C_0\tan\frac{z}{a} + C_1}{\sqrt{R(z)}}.$$

3) Case of discontinuous coefficients

Assume L_0 to be a closed contour as in Section 1, but $G(t)$ and $g(t)$ have a finite number of discontinuities $c_1, \cdots, c_p$ of the first kind on L_0, belong to $H(A, \mu)$ on each arc of L_0 (arc between two adjacent discontinuities) including its end-points (the values at these points are understood by the one sided limits) and $G(t) \neq 0$.

1.3. *Periodic Riemann-Hilbert Boundary Value Problems: Case of the Half-plane*

1) Formulation of the problems

Assume L_k is the segment from $-\frac{1}{2}a\pi + \frac{1}{2}ka\pi$ to $\frac{1}{2}a\pi + \frac{1}{2}ka\pi$ $(k = 0, \pm 1, \pm 2, \cdots)$ lying on the real axis with length $a\pi$ $(a > 0)$. Then $|L_k|$ is a set of periodic segments with period $a\pi$, the union of which is the real axis. Denote the lower half-plane by S^- and the upper, by S^+.

The Riemann-Hilbert boundary value problem of the half-plane is to find a function $w(z) \equiv u - iv$ holomorphic in S^- with period $a\pi$, satisfying the boundary conditions

$$a(x)u + b(x)v = F(x), \;\; x \in L_k, k = 0, \pm 1, \pm 2, \cdots,$$

where $a(x), b(x)$ and $F(x)$ are given functions with period $a\pi$, arcwise with Hölder-Lipschitz continuity, i.e., having a finite number of discontinuities on each $L_k \in H(A, \mu)$ on each continuous closed interval between two adjacent discontinuous points(called nodes), and $a(x), b(x)$ have no common zeros for any x. Without loss of generality, we may always assume that all of them are continuous at $x = \pm\frac{1}{2}a\pi$(and their congruent points).

2) An important particular case

We would give solutions for a particular but very important case which often occur in practice and would be met later in this chapter.

Let $\gamma_0 = [-l, l]$, $0 < l < \frac{a\pi}{2}$, and $\gamma_0' = L_0 - \gamma_0$. Denote the set of all the segments congruent to γ_0 and γ_0'(mod $a\pi$) by γ and γ' respectively. In the particular case considered hereby:

$$a(x) + ib(x) = \begin{cases} a_1 + ib_1 & x \in \gamma, \\ a_2 + ib_2 & x \in \gamma'. \end{cases} \tag{1.3}$$

where a_j, b_j are real constants with $a_j + ib_j \neq 0$ $(j = 1, 2)$ and $a_1 + ib_1 \neq a_1 + ib_2$.

A particular solution is well-known as

$$\Omega_1(z) = \frac{X(z)}{a\pi i} \int_{L_0} \frac{F(t)}{X^+(t)[a(t) - ib(t)]} \cot \frac{t - z}{a} dt, \mathrm{Im} z \neq 0.$$

2. Periodic Problems for Isotropic Material in Elastic Theory

There were many works in literature on the periodic problems in the theory of plane elasticity for isotropic material, e.g., researches on stress distribution in the neighborhoods of periodic holes by R. C. J. Howland, G. N. Savin, L. M. Tang, M. Isida, study on periodic contact problems by G. M. L. Gladwall and investigation for expression of periodic stress functions by S. Morigashi. All these works have limitations in either the shape of the holes or the boundary conditions, and the discussions are incomplete. Especially, there were few studies on the possibility of quasi-periodic displacements. see Ref.1-2,8,13,15-18,20-22.

2.1. *Stress Functions*

1) General expression of stress functions

Let the elastic plane possess a row of periodic holes with boundaries $L_j, j = 0, \pm 1, \pm 2, \cdots$,, where L_j consists of piecewise smooth closed contours $l_k^{(j)}, k = 1, \cdots, n$ and for the same k $l_k^{(j)}, k = 1, \cdots, n$ are periodically arranged. l_k^0 will be denoted briefly by l_k. All the contours are oriented counter-clockwise. The region occupied by the elastic body is denoted as S^-, the region bounded by $l_k^{(j)}$ as $S_k^{(j)+}$, and $S_k^{(0)+}$ briefly as S_k^+. Denote the strip region $|x| < \frac{1}{2}a\pi$ by S_0. Let $S_0^+ = \sum_{k=1}^{n} S_k^+$ and $S_0^+ = S_0 - S_0^+$.

Denote the stresses at any point $z = x + iy$ in S^- by $\sigma_x(z)$, $\sigma_y(z)$, $\tau_{xy}(z)$ and the (complex) displacement by $D(x) = u + iv$. It is well-known in Ref.23, that they may be expressed in terms of (complex) stress functions $\varphi(z)$ and $\psi(z)$or their derivatives $\Phi(z) = \varphi'(z)$and $\Psi(z) = \psi'(z)$ as follows:

$$\begin{cases} \sigma_x + \sigma_y = 2[\Phi(z) + \overline{\Phi(z)}], \\ \sigma_y - \sigma_x + 2i\tau_{xy} = 2[\overline{z}\Phi'(z) + \Psi(z)], \\ 2\mu d = 2\mu(u + iv) = \kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}, \end{cases} \tag{2.1}$$

where μ is the shearing modulus of the elastic medium, κ is a constant related to its Passion ratio $\sigma(1 < \kappa < 3)$ and $\varphi(z)$, $\psi(z)$ are functions analytic (in general, multi-valued) functions in S^- with their derivatives holomorphic, i.e., single-valued, in S^-.

We always assume that the stresses are periodic and bounded at $z = \pm\infty i$.

Theorem 3 In the isotropic infinite elastic plane weakened by a row of holes of any shape with period $a\pi(a > 0)$, if the stresses are known to be periodic and bounded at infinity , then the relative displacements must be quasi-periodic , i.e.,

$$\begin{cases} 2\mu[u(z + a\pi) - u(z)] = a\pi q \\ v(z + a\pi) - v(z) = 0 \end{cases} \tag{2.2}$$

where q is a certain real constant.

$a\pi q/2p$ is called the addendum of the quasi-periodic function $u(z)$. By the previous expression, we know that

$$\gamma = a\pi(\kappa\beta - q).$$

Theorem 4 Suppose there is a row of holes with period $a\pi$ $(a > 0)$ in the isotropic elastic plane, the boundary of which is $L_j, j = 0, \pm 1, \pm 2, \cdots$, where each L_j consists of n piecewise smooth closed contours $l_k^{(j)}, k = 1, 2, \cdots n$, and for a fixed $l_k^{(j)}, j = 0, \pm 1, \pm 2, \cdots$, are arranged periodically. Assume the stresses are periodic, bounded at $z = \pm\infty i$ and the principal vector of the external stresses on $l_k^{(0)}$ is $X_k + iY_k$. Then the stress functions and $\varphi(z)$ and $\psi(z)$ respectively have the expressions:

$$\varphi(z) = -\frac{1}{2\pi(\kappa+1)}(X_k + iY_k)\log\sin\frac{z-z_k}{a} + \beta z + \varphi_0(z) \tag{2.3}$$

$$\begin{aligned}\psi(z) = &\frac{\kappa}{2\pi(\kappa+1)}\sum_{k=1}^{n}(X_k - iY_k)\log\sin\frac{z-z_k}{a}\\ &+ \frac{\kappa}{2a\pi(\kappa+1)}\sum_{k=1}^{n}(X_k - iY_k)\cot\frac{z-z_k}{a}\\ &+ (\kappa\beta - \beta + q)z - z\varphi_0'(z) + \psi_0(z).\end{aligned} \tag{2.4}$$

where $\varphi_0(z)$ and $\psi_0(z)$ are functions holomorphic and $a\pi$-periodic in the elastic region S^-.

Corollary 4.1 Assume the elastic body as above. If the resultant of the principal vectors on the boundaries of the holes in a periodic strip is zero and the stresses at $z = \pm\infty\ i$ are $\sigma = \sigma_y(\pm\infty i)$ and $\tau = \tau_{xy}(\pm\infty i)$, then both the stress functions $\varphi(z)$and $\psi(z)$ are single-valued, and

$$\varphi(z) = \varphi_0(z) + \beta z,$$

$$\psi(z) = \psi_0(z) - z\varphi'(z) + \kappa\bar{\beta}z = (\kappa\bar{\beta} - \beta)z - z\varphi_0'(z) + \psi_0(z),$$

where $\beta = \frac{-\sigma + i\tau}{1+\kappa}$, while both $\varphi_0(z)$ and $\psi_0(z)$ are $a\pi$-periodic.

Theorem 5 If the displacements in an isotropic plane elastic body are quasi-periodic, then the stresses must be periodic.

2) Formulation of the fundamental problem

First fundamental problem Given the periodic stress function $X_n(t)+iY_n(t)$ on the boundary L of the elastic region and the stresses at $z = -\infty i$(or $z = +\infty i$), find the elastic equilibrium (i.e., the stress distribution in the elastic body).

This is the most general formulation of the problem under the assumption of stresses to be periodic and bounded at infinity. In this case, the stresses at $z = +\infty i$ (or$z = -\infty i$) is also known, and so do β and q.

The problem may be reduced to the following boundary value problem:

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = f(t) + C(t),\ t \in L,$$

where $C(t)$ represents a step function on different boundary contour with different value and (we have assumed $z = 0 \in S^-$);

$$f(t) = -i\int_0^t (X_n + iY_n)ds, \quad t \in L,$$

where s is the arc-length parameter on boundary contour;

$$\varphi_0(t) + (t - \bar{t})\overline{\varphi_0'(t)} + \overline{\psi_0(t)} = f_0(t) + C(t),$$

where we have put

$$\begin{aligned} f_0(t) = & \frac{1}{2\pi(\kappa+1)} \sum_{k=1}^{n} (X_k + iY_k) \log \sin \frac{t - z_k}{a} \\ & - \frac{\kappa}{2\pi(\kappa+1)} \sum_{k=1}^{n} (X_k + iY_k) \log \sin \frac{t - z_k}{a} \\ & + \frac{t - \bar{t}}{2a\pi(\kappa+1)} \sum_{k=1}^{n} (X_k + iY_k) \cot \frac{t - z_k}{a} \\ & - (\beta + \bar{\beta})t - (\kappa\beta - \bar{\beta} + q)\bar{t} + f(t), \end{aligned}$$

which is singer-valued on L obviously.

Second fundamental problem Given the relative displacements on the boundary contours of the periodic holes, the resultant principal vector $X + iY$ of the external stresses along the boundary contours in a periodic strip and the stresses at, $z = -\infty i$(or $z = +\infty i$), find the equilibrium.

Here, the relative displacements mean that they are quasi-periodic but the constant q is not given. As in the first fundamental problem, this is the most general formulation of the problem. It may be reduced to solve a Fredholm integral equation as well.

For convenience of discussion, in the sequel, we always assume that the displacements are also periodic.

3) Stress functions for elastic half-plane

Assume the elastic body occupies the lower half-plane S^-. In this case, the stresses and displacements may be expressed in terms of a single stress function. When z lies in the upper half-plane S^+, let

$$\bar{\Phi}(z) = \overline{\Phi(\bar{z})}, \ \bar{\Psi}(z) = \overline{\Psi(\bar{z})}, \ z \in S^+,$$

where $\Phi(z) = \varphi'(z), \Psi(z) = \psi'(z)$ are the stress functions.

Theorem 6 In the periodic problems for isotropic elastic half-plane, $\Phi(z)$ is $a\pi$-periodic for $z \in S^+$ and $z \in S^-$ including its boundary values; and is bounded at $z = \pm\infty i$.

2.2. *Periodic Fundamental Problems of Elastic Half-plane*

1) The first fundamental problem

Let the isotropic elastic body occupy the lower half-plane S^- in the z-plane. On the x-axis, denote $z = t(t$ is real). Given the external stresses on the x-axis:

$$\sigma_y(t) = -P(t), \quad \tau_{xy}(t) = T(t),$$

being arcwise $\in H(A,\mu)$ and with period $a\pi$. Here P means the normal pressure distribution. Assume again both the stresses and displacements are periodic and the stresses at $z=-\infty i$ are bounded. Find the stress distribution and the displacements on the whole elastic body, called the periodic first fundamental problem of the half-plane.

We have

Theorem 7 Under the above assumptions, the solution of the periodic first fundamental problem uniquely exists.

The expression of $\Phi(z)$ can be written as

$$\Phi(z)=\frac{1}{2a\pi i}\int_{L_0}[P(t)+iT(t)]\cot\frac{t-z}{a}dt+\frac{1}{2}\frac{\kappa-1}{\kappa+1}P^* \tag{2.5}$$

P^* and T^* have clear mechanical meaning. and so

$$\sigma_y(-\infty\ i)=P^*,\tau_{xy}(-\infty\ i)=T^*, \tag{2.6}$$

$$\sigma_x(-\infty\ i)=-\frac{3-\kappa}{1+\kappa}P^*. \tag{2.7}$$

Corollary 7.1 For the periodic first fundamental problem of isotropic elastic half-plane, if the resultant of the external normal stresses on the boundary of a period is a pressure force:

$$\int_{L_0}P(t)dt=a\pi P^*>0,$$

then σ_x is a compression at $z=-\infty i$, given by (2.7).

2) The second fundamental problem

Assume that, on the boundary L of the isotropic elastic half plane S^-, the complex displacement $u^-+iv^-=g(t)$ is given, where $g(t)$ is continuous and $a\pi$-periodic (up to an arbitrary constant term, corresponding to a rigid translation of the whole elastic body), $g'(t)\in H(A,\mu)$ arcwise, and the principal vector $X+iY$ of the external stresses on a period of the boundary is also given. Again we assume the stresses and the displacements are periodic and the stresses at $z=-\infty i$are bounded. Find the elastic equilibrium here is the periodic second fundamental problem of the half-plane.

Theorem 8 Under the above assumptions, the solution of the periodic second fundamental problem of the half-plane uniquely exists.

We obtain

$$\Phi(z)=\begin{cases}\frac{\mu}{a\pi i}\int_{L_0}g'(t)\cot\frac{t-z}{a}dt-\frac{\kappa(Y-iY)}{(\kappa+1)a\pi},\ z\in S^+\\ -\frac{\mu}{\kappa a\pi i}\int_{L_0}g'(t)\cot\frac{t-z}{a}dt+\frac{(Y-iX)}{(\kappa+1)a\pi},\ z\in S^-\end{cases} \tag{2.8}$$

The following two corollaries are evident.

Corollary 8.1 $\sigma_x(-\infty i)=0$ iff $Y=0$.

Corollary 8.2 The horizontal displacement u is bounded when and only when $X = 0$ and so does the vertical displacement v when and only when $Y = 0$.

3) The mixed fundamental problem

Let the elastic half-plane S^- as before. On the boundary L_0 in a period, given displacement (up to a constant term) on its sub-segment $\gamma_0(-l \leq t \leq l)$

$$u^- + iv^- = g(t),\ t \in \gamma_0$$

where $g(t)$ is continuous with $g'(t)$ and on $\gamma_0' = L_0 - \gamma_0$, given the external stresses for briefness, to be zero:

$$\sigma_y^-(t) = \tau_{xy}^-(t) = 0,\ l \leq |t| \leq \frac{1}{2}a\pi.$$

Besides, the principal vector $X + iY$ of the external stresses on γ_0 is also given. All the conditions given are $a\pi$-periodic. Again assume the stresses and the displacements are periodic and the stresses at $z = -\infty i$ are bounded. Find the elastic equilibrium. The problem is called the periodic mixed fundamental problem.

Theorem 9 Under the above conditions, the solution of the periodic mixed fundamental problem uniquely exists. We have

$$\Phi(z) = \frac{2\mu\varepsilon i}{\kappa + 1}\left\{1 - \frac{X(z)}{\cos\frac{l}{a}}(\cosh A \tan\frac{z}{a} - i\sinh A)\right\}. \tag{2.9}$$

2.3. *Periodic Contact Problems*

1) The case without friction

Assume a series of periodic stamps (of the same shape) are pressed on the boundary of an isotropic elastic half-plane S^-. In the present paragraph, it assumed that there exists no friction between the stamps and the half-plane. Out of the stamps, the condition subjected to the boundary of the half-plane is $\sigma_x = \tau_{xy} = 0$. Right beneath the bases of the stamps, only the periodic vertical displacement $v(t)$ is given while the horizontal displacement $u(t)$ is unknown. On the boundaryL_0, let the interval pressed be $\gamma_0 : -l \leq t \leq l$. Since there is no friction $\tau_{xy} = 0$ also on γ_0 but σ_y is unknown . Besides, assume the load applied to each stamp is a positive pressure force P_0, i.e. $Y = -P_0$ and $X = 0$. Find the elastic equilibrium.

Let $y = f(x)$ be the equation of the vase of the stamp pressed on S^- and $f'(x) \in H(A, \mu)$. Thus the boundary conditions on L_0 are:

$$\begin{aligned}&\tau_{xy}^-(t) = 0,\ t \in L_0;\ \sigma_y^-(t) = 0,\ t \in L_0 - \gamma_0\\ &v(t) = f(t),\ t \in \gamma_0;\end{aligned}$$

and the principal vector of the external stresses on γ_0 is $X + iY = -P_0 i$.
Under these boundary conditions, by the principle of equilibrium, we know that

$$\sigma = \sigma_y(-\infty i) = -\frac{P_0}{a\pi},\ \tau = \tau_{xy}(-\infty i) = 0. \tag{2.10}$$

Note that $\Phi(z)$ is holomorphic in the plane cut by γ (γ_0 and its periodic congruent),

$$\Phi(z) = \frac{2\mu}{(\kappa+1)a\pi\sqrt{R(z)}}\int_{\gamma_0} f'(t)\sqrt{R(t)}\cot\frac{t-z}{a}dt + \frac{\beta_0\tan\frac{z}{z}+\beta_1}{\sqrt{R(z)}} + \beta_2, \quad (2.11)$$

where we have written $X(z)$ as

$$X(z) = \frac{1}{i\sqrt{R(z)}},\ R(z) = \tan^2\frac{l}{a} - \tan^2\frac{x}{a} \quad (2.12)$$

where $\sqrt{R(z)}$ takes positive value $\sqrt{R(t)}$, when z tends to $t \in \gamma_0$ from S^+. It is also seen that by (2.12),

$$\lim_{z\to\pm\infty i}(z-\overline{z})\Phi'(z) = 0 \quad (2.13)$$

remains valid.

From the periodic condition of the displacements and the condition of equilibrium of the stresses at $z = \pm\infty i$, we get

$$\beta_0 = -\frac{2\mu}{(\kappa+1)a\pi}\int_{\gamma_0} f'(t)\sqrt{R(t)}dt,$$

$$\beta_1 = \frac{P_0}{2a\pi\cos\frac{l}{a}},$$

$$\beta_1 = \frac{\kappa-1}{\kappa+1}\frac{P_0}{2a\pi}.$$

The required unique solution is finally obtained

$$\Phi(z) = \frac{2\mu}{(\kappa+1)a\pi\sqrt{R(z)}}\int_{\gamma_0} f'(t)\sqrt{R(t)}(\cot\frac{t-z}{a} - \tan\frac{z}{a})dt + \frac{P_0}{2a\pi\cos\frac{l}{a}\sqrt{R(z)}} + \frac{\kappa-1}{\kappa+1}\frac{P_0}{2a\pi} \quad (2.14)$$

The pressure distribution right beneath the bases of the stamps could be easily evaluated by

$$P(t_0) = \frac{4\mu}{(\kappa+1)a\pi\sqrt{R(t_0)}}\int_{-l}^{l} f'(t)\sqrt{R(t)}(\cot\frac{t-t_0}{a} - \tan\frac{t_0}{a})dt + \frac{P_0}{a\pi\cos\frac{l}{a}\sqrt{R(t_0)}} \quad (2.15)$$

Example Periodic stamps with horizontal rectilinear base.
Here $f'(t) = 0$, by (2.14) and (2.15):

$$\Phi(z) = \frac{P_0}{2a\pi\cos\frac{l}{a}\sqrt{\tan^2\frac{l}{a} - \tan^2\frac{z}{a}}} + \frac{\kappa-1}{\kappa+1}\frac{P_0}{2a\pi}$$
$$= \frac{P_0\cos\frac{l}{a}}{2a\pi\sqrt{\sin\frac{l+z}{a}\sin\frac{l-z}{a}}} + \frac{\kappa-1}{\kappa+1}\frac{P_0}{2a\pi},$$

$$P(t) = \frac{P_0}{a\pi \cos\frac{l}{a}\sqrt{\tan^2\frac{l}{a} - \tan^2\frac{t}{a}}} = \frac{P_0}{a\pi\sqrt{\sin\frac{l+t}{a}\sin\frac{l-t}{a}}},$$

where the radical involved is taken as the branch, when the plane is cut by γ taking positive value as $z \to t \in \gamma_0$ from S^+.

2) The case with friction

Now assume the friction coefficient $k \neq 0$ between the periodic stamps and eh elastic half-plane, that means, beneath the stamps, between the shearing stress $T(t) = \tau_{xy}(t)$ and the normal pressure $P(t) = -\sigma_y(t)$, there exist the relation

$$T(t) = kP(t), \ t \in \gamma_0. \tag{2.16}$$

Again assume $v^-(t) = f(t)$ on γ_0 with $f(t) \in H(A,\mu)$ and the external pressure force P_0 are given on γ_0. The principal vector of the external stresses on γ_0 is known as $X + iY = T_0 - iP_0 = (k-i)P$. On $\gamma_0' = L_0 - \gamma_0$, $T(t) = P(t) = 0$.

We obtain the general solution

$$\Phi(z) = \frac{2\mu(1+ik)e^{\pi a i}\cos\pi a X(z)}{a\pi(\kappa+1)}\int_{\gamma_0}\frac{f'(t)}{X^+(t)}\cot\frac{t-z}{a}dt + X(z)(1+ik)i(\beta_0\tan\frac{z}{a} + \beta_1) + \beta_2. \tag{2.17}$$

From the condition of periodicity of the displacements and the condition of equilibrium of the stresses at $z = -\infty i$, we get

$$\begin{cases} \beta_0 = -\dfrac{2\mu\cos\pi a}{(\kappa+1)a\pi}\displaystyle\int_{\gamma_0} f'(t)Q(t)dt + \dfrac{P_0\sin(1-\lambda)\pi a}{2a\pi\cos\pi a\cos\frac{l}{a}\sqrt{R(z)}} \\ \beta_1 = \dfrac{P_0\cos(1-\lambda)\pi a}{2a\pi\cos\pi a\cos\frac{l}{a}} \\ \beta_2 = \dfrac{\kappa-1}{\kappa+1}\dfrac{(k^2+1)P_0}{2a\pi} \end{cases} \tag{2.18}$$

and

$$P(t_0) = \frac{2\mu\sin 2\pi a}{(\kappa+1)}f'(t_0) + \frac{4\mu\cos^2\pi a}{a\pi(\kappa+1)Q(t_0)}\int_{-l}^{l} f'(t)Q(t)\cot\frac{t-t_0}{a}dt + \frac{2\cos\pi a}{Q(t_0)}(\beta_0\tan\frac{t_0}{a} + \beta_1), \ t \in \gamma_0. \tag{2.19}$$

Example Periodic stamps with horizontal rectilinear base.

In this case, $f'(t) = 0$.

$$\Phi(z) = \frac{P_0(1+ik)e^{\pi a i}\cos[\frac{z}{a}(1-\lambda)\pi a]}{2a\pi\cos\pi a\sin^{\frac{1}{2}+a}\frac{l+z}{a}\sin^{\frac{1}{2}-a}\frac{l-z}{a}} + \frac{\kappa-1}{\kappa+1}\frac{(k^2+1)P_0}{2a\pi} \tag{2.20}$$

$$P(t) = \frac{P_0 \cos[\frac{t}{a}(1-\lambda)\pi a]}{a\pi \sin^{\frac{1}{2}+a} \frac{l+t}{a} \sin^{\frac{1}{2}-a} \frac{l-t}{a}}, \ t \in \gamma_0. \tag{2.21}$$

3. Periodic Problems for Anisotropic Medium

3.1. *The stress Functions*

1) Basic assumptions

All the discussion below are under the following assumption, the medium of the $a\pi$-periodic region is anisotropic, the stresses and displacements are periodic and the stresses at infinity are bounded, and the involved boundary conditions are periodic. Thus, we need only restrict our discussions in a period part of the elastic body. Moreover, we assume the elastic body occupies the lower half-plane in the z-plane ($z = x + iy$) and so there is only one point at infinity $z = -\infty i$.

The principal vector of $X(-\infty i) + iY(-\infty i)$ the external stresses at $z = -\infty i$ is understood by the limit of the principal vector of the stresses along a line-segment from z to $z + a\pi$ in S^- as $z \to -\infty i$ i.e.

$$X(-\infty i) = a\pi\tau_{xy}(-\infty i), \ Y(-\infty i) = a\pi\sigma_y(-\infty i).$$

If the principal vector of the external stresses on the boundary in a period is $X + iY$, then by the condition of equilibrium,

$$x + iY = X(-\infty i) + iY(-\infty i)$$

we have

$$\sigma_y(-\infty i) = \frac{Y}{a\pi}, \tau_{xy}(-\infty i) = \frac{X}{a\pi}. \tag{3.1}$$

2) Periodicity of the stress functions for anisotropic medium

For anisotropic elastic body, the stress components $\sigma_x, \sigma_y, \tau_{xy}$ and displacement components u, v may be expressed by means of $\varphi(z_1)$ and $\psi(z_2)$ or their derivatives $\Phi(z_1) = \varphi'(z_1)$ and $\Psi(z_2) = \psi'(z_2)$ (stress functions):

$$\sigma_x = \mu_1^2\Phi(z_1) + \overline{\mu}_1^2\overline{\Phi(z_1)} + \mu_2^2\Psi(z_2) + \overline{\mu}_2^2\overline{\Psi(z_2)} \tag{3.2}$$

$$\sigma_y = \Phi(z_1) + \overline{\Phi(z_1)} + \Psi(z_2) + \overline{\Psi(z_2)} \tag{3.3}$$

$$\tau_{xy} = -[\mu_1\Phi(z_1) + \overline{\mu}_1\overline{\Phi(z_1)} + \mu_2\Psi(z_2) + \overline{\mu}_2\overline{\Psi(z_2)}] \tag{3.4}$$

$$u = p_1\varphi(z_1) + \overline{p}_1\overline{\varphi(z_1)} + p_2\psi(z_2) + \overline{p}_2\overline{\psi(z_2)} \tag{3.5}$$

$$u = q_1\varphi(z_1) + \overline{q}_1\overline{\varphi(z_1)} + q_2\psi(z_2) + \overline{q}_2\overline{\psi(z_2)} \tag{3.6}$$

where $\varphi(z_1)$ and $\psi(z_2)$ are functions holomorphic in z_1 and z_2 respectively, in which,

$$z_1 = x + \mu_1 y, \ z_2 = x + \mu_2 y,$$

and

$$\begin{cases} p_1 = \beta_{11}\mu_1^2 + \beta_{12} - \beta_{16}\mu_1, \\ p_2 = \beta_{11}\mu_2^2 + \beta_{12} - \beta_{16}\mu_2, \\ q_1 = \dfrac{\beta_{12}\mu_1^2 + \beta_{22} - \beta_{26}\mu_1}{\mu_1}, \\ q_2 = \dfrac{\beta_{12}\mu_2^2 + \beta_{22} - \beta_{26}\mu_2}{\mu_2}, \end{cases} \tag{3.7}$$

and $\mu_1, \overline{\mu}_1, \mu_2, \overline{\mu}_2$ are the root of the equation

$$\beta_{11}s^4 - 2\beta_{16}s^3 + (2\beta_{11} + \beta_{66})s^2 - 2\beta_{26}s + \beta_{22} = 0. \tag{3.8}$$

while

$$\begin{matrix} \beta_{11} & \beta_{12} & \beta_{16} \\ \beta_{12} & \beta_{22} & \beta_{26} \\ \beta_{16} & \beta_{26} & \beta_{66} \end{matrix}$$

are the elastic coefficients of the anisotropic elastic body.

Lemma 1 Under the basic assumptions, the stress functions $\Phi(z_1)$ and $\Psi(z_2)$ are $a\pi$- periodic functions.

3.2. *Periodic Fundamental Problems of Anisotropic Half-plane*

1) The first fundamental problem

Assume the anisotropic body occupies the lower half-plane S^- of the z-plane. Denote $z = t$ (real) on the x-axis. Given the external stresses on the x-axis:

$$\sigma_y(t) = -P(t), \ \tau_{xy}(t) = T(t), \tag{3.9}$$

which are arcwise Hölder continuous and periodic. Under the basic assumptions, find the stress distribution and displacements, called the periodic first fundamental problem.

Theorem 10 Under the above assumptions, the first fundamental problem of the anisotropic half-plane is uniquely solvable.
We get stress functions

$$\Phi(z_1) = \frac{-1}{(\mu_1 - \mu_2)2a\pi i}\int_{L_0}[\mu_2 P(t) - T(t)]\cot\frac{t - z_1}{a}dt + \gamma_1, \tag{3.10}$$

$$\Psi(z_2) = \frac{1}{(\mu_1 - \mu_2)2a\pi i}\int_{L_0}[\mu_1 P(t) - T(t)]\cot\frac{t - z_2}{a}dt + \gamma_2, \tag{3.11}$$

where

$$\begin{aligned} \gamma_1 = {} & -\frac{1}{2}\frac{1}{\mu_1 - \mu_2}[(\mu_2 - \mu_1)\Phi(-\infty i) \\ & + (\mu_2 - \mu_1)\overline{\Phi(-\infty i)} + (\mu_2 - \overline{\mu_2})\Psi(-\infty i)], \end{aligned} \tag{3.12}$$

$$\gamma_2 = \frac{1}{2}\frac{1}{\mu_1 - \mu_2}[(\mu_1 - \mu_2)\Psi(-\infty i)\Phi(-\infty i) + (\mu_1 - \overline{\mu_1})\overline{\Phi(-\infty i)} + (\mu_1 - \overline{\mu_2})\overline{\Psi(-\infty i)}]. \quad (3.13)$$

In order to determine γ_1 and γ_2 by noting (3.12) and (3.13), it is easily seen that

$$Re|\gamma_1 + \gamma_2| = 0, \quad (3.14)$$

$$Re|\mu_1\gamma_1 + \mu_2\gamma_2| = 0. \quad (3.15)$$

2) The second fundamental problem

Assume that on the boundary x-axis of the anisotropic elastic half-plane S^-, the displacement

$$u^- + iv^- = g_1(t) + ig_2(t) \quad (3.16)$$

is given, where $g_1(t) + ig_2(t)$ is continuous and $g_1'(t) + ig_2'(t)$ is Hölder continuous arcwise. Moreover, on the segment L_0 of the boundary in a period, the principal vector $X + iY$ of the external stresses is also given. Under these basic boundary conditions, find the equilibrium, called the periodic second fundamental problem.

Theorem 11 Under the above assumptions, the solution of the second fundamental problem of the uniquely exists.

We get stress functions

$$\Phi(z_1) = \frac{1}{q_1p_2 - p_1q_2}\frac{1}{2a\pi i}\int_{L_0}[q_2g_1' - p_2g_2')]\cot\frac{t - z_1}{a}dt + \gamma_1 \quad (3.17)$$

$$\Psi(z_2) = \frac{-1}{(q_1p_2 - p_1q_2)}\frac{1}{2a\pi i}\int_{L_0}[q_1g_1' - p_1g_2')]\cot\frac{t - z_2}{a}dt + \gamma_2 \quad (3.18)$$

where γ_1 and γ_2 may be obtained by solving

$$Re[p_1\gamma_1 + p_2\gamma_2] = 0,$$

$$Re|q_1\gamma_1 + q_2\gamma_2| = 0,$$

$$Re[\gamma_1 + \gamma_2] = \frac{Y}{2a\pi},$$

$$Re[\mu_1\gamma_1 + \mu_2\gamma_2] = -\frac{X}{2a\pi}.$$

3.3. *Periodic Contact Problem for Anisotropic Medium*

1) Formulation of the problem and the solutions

Formulation of the periodic contact problem in anisotropic half-plane S^- is as follows.

Assume that a row of periodic stamps (with bases of the same shape) pressed on S^- and there exists friction between the stamps and with coefficient of friction ρ, that is, beneath the stamps, the shearing stress $T(x) = \tau_{xy}(x)$ and normal pressure $P(x) = -\sigma_y(x)$ obey the Coulomb's law:

$$T(x) = \rho P(x),\ x \in \gamma_0,$$

or

$$\tau_{xy}(x) + \sigma_y(x) = 0,\ x \in \gamma_0,$$

where $\gamma_0 : -l_0 \le x \le l_0$, is the contact line segment in $L_0 : -\frac{1}{2}a\pi < x < \frac{1}{2}a\pi$. Assume the stamps are in the limiting situation of equilibrium. On the free interval, there is no load, i.e.

$$\sigma_y = 0,\ \tau_{xy} = 0.$$

Besides, assume the vertical displacement

$$v^-(\dot{x}) = f(x) \tag{3.19}$$

is given, where $y = f(x)$ is the equation of the bases of the stamps, which is $a\pi$-periodic with $f'(x) \in H(A, \mu)$. Moreover, the external pressure force P_0 applied on each stamp is also given and so the principal; vector of the external stresses is $X + iY = T_0 - iP_0 = (\rho - i)P_0$. Find the elastic equilibrium under these assumptions.

$$\begin{cases} \sigma_y(x) = 0,\ \tau_{xy}(x) = 0,\ x \in L_0 - \gamma_0, \\ \tau_{xy}(x) + \rho\sigma_y(x) = 0,\ v^-(x) = f(x),\ x \in \gamma_0. \end{cases} \tag{3.20}$$

In order to solve the problem, we have to transform (3.1). Introduce two functions represented by integrals with Hilbert kernel:

$$w_1(z) = u_1 - iv_1 = \int_{L_0} \sigma_y(t) \cot \frac{t-z}{a} dt, \tag{3.21}$$

$$w_2(z) = u_2 - iv_2 = \int_{L_0} \tau_{xy}(t) \cot \frac{t-z}{a} dt + \beta. \tag{3.22}$$

Therefore, we obtain

$$\begin{aligned} w_1(z) = \frac{\pm ie^{\pm\theta\pi i} \cos\theta\pi E(z)}{A_3 - \rho A_4} \int_{\gamma_0} \frac{f'(t)}{E(t)} \cot \frac{t-z}{a} \\ \pm ie^{\pm\omega_1 i}(C_1 \tan \frac{z}{a} + C_2)E(z), \quad z \in S^{\pm}, \end{aligned} \tag{3.23}$$

and our solution is $w_1(z)$ when $z \in S^-$.

As for $w_2(z)$, by boundary condition, it is evident that

$$w_2(z) = -\rho w_1(z) + \beta. \tag{3.24}$$

From periodic condition for displacements and equilibrium condition at $z = -\infty\ i$, the real conditions C_1, C_2 and β may be obtained by solving

$$\begin{aligned}
&\left\{(\varepsilon_1 + i\delta_1)e^{-i\frac{2l\theta}{a}} + (\varepsilon_2 + i\delta_2)e^{i\frac{2l\theta}{a}}\right\} C_1 \\
&+ i\left\{(\varepsilon_1 + i\delta_1)e^{-i\frac{2l\theta}{a}} - (\varepsilon_2 + i\delta_2)e^{i\frac{2l\theta}{a}}\right\} C_2 + (\varepsilon_3 + i\delta_3)\beta a\pi \\
&= \frac{\cos\pi\theta}{A_3 - \rho A_4}\left\{(\varepsilon_1 + i\delta_1)e^{-i\frac{2l\theta}{a}} + (\varepsilon_2 + i\delta_2)e^{i\frac{2l\theta}{a}}\right\}\int_{\gamma_0}\frac{f'(t)}{E(t)}dt \\
&\frac{P_0}{\cos\frac{l}{a}} - \frac{\cos\pi\theta\sin\frac{2l\theta}{a}}{A_3 - \rho A_4}\int_{\gamma_0}\frac{f'(t)}{E(t)}dt = C_2\cos\frac{2l\theta}{a} - C_1\sin\frac{2l\theta}{a}.
\end{aligned}$$

2) The pressure beneath the stamps

$$\begin{aligned}
p(x) = &\frac{1}{2(A_3 - \rho A_4)}\left\{-\sin 2\theta\pi f'(x) + \frac{2\cos^2\theta\pi E(x)}{a\pi}\int_{\gamma_0}\frac{f'(t)}{E(t)}\cot\frac{t-x}{a}dt\right\} \\
&+ \frac{\cos\theta\pi}{a\pi}E(x)(C_1\tan\frac{x}{a} + C_2), x \in \gamma_0.
\end{aligned}$$

Example Consider the case where the stamps possess periodic horizontal rectilinear bases, here, $f'(x) = 0$. Then

$$p(x) = \frac{\cos\theta\pi}{a\pi}E(x)(C_1\tan\frac{x}{a} + C_2),\ x \in \gamma_0,$$

where C_1 and C_2 are determined by

$$\left\{(\varepsilon_1 + i\delta_1)e^{-i\frac{2l\theta}{a}} + (\varepsilon_2 + i\delta_2)e^{i\frac{2l\theta}{a}}\right\} C_1$$

$$+i\left\{(\varepsilon_1 + i\delta_1)e^{-i\frac{2l\theta}{a}} - (\varepsilon_2 + i\delta_2)e^{i\frac{2l\theta}{a}}\right\} C_2 + (\varepsilon_3 + i\delta_3)\beta a\pi = 0$$

$$C_2\cos\frac{2l\theta}{a} - C_1\sin\frac{2l\theta}{a} = \frac{P_0}{\cos\frac{l}{a}}.$$

See Ref.3-7,14,19.

4. Periodic Crack Problems in Plane Elasticity

For isotropic elastic plane weakened by a periodic row of cracks, W.T.Koiter had studied the first fundamental problems by complex variable methods for the cases where the cracks are rectilinear and collinear (in the direction of period), or parallel and perpendicular to the direction of period, under very special assumptions for the external stresses subjected on the cracks as well as those at infinity. See Ref.6-7,14,17,19,28,30.

4.1. *Fundamental Problems of Isotropic Plane with Periodic Collinear Cracks*

1) General comments

Consider the isotropic elastic infinite plane weakened by periodic rectilinear cracks with the same direction as the period $a\pi$. without loss of generality, we may ask that they are situated on the real axis.

Assume that there are n cracks in the periodic strip $|x| < \frac{1}{2}a\pi$, namely, l_k : $a_k \leq t \leq b_k(a_{k+1} > b_k)$, $k = 1, ..., n-1$, positively oriented from a_k to b_k. denote $l_0 = \sum_{k=1}^{n} l_k$, and the principal vector of the external stresses on l_k by $X_k + iY_k$, The elastic region is denoted by S. Other notations are the same as in previous section.

The following discussions are made under the assumptions that the stresses are periodic and bounded at $z = \pm\infty i$ while the displacements are quasi-periodic.

Introduce functions

$$\left.\begin{aligned} w(z) &= z\bar{\Phi}(z) + \bar{\Psi}(z), \\ \Omega(z) &= w'(z) = \bar{\Phi}(z) + z\bar{\Phi}'(z) + \bar{\Psi}'(z), \end{aligned}\right\} z \in S, \tag{4.1}$$

where $\Phi(z), \Psi(z)$ are complex stress functions and $\bar{\Phi}(z) = \overline{\Phi(\bar{z})}, \bar{\Psi}(z) = \overline{\Psi(\bar{z})}$. It is easily seen

$$\sigma_y - i\tau_{xy} = \Phi(z) + \Omega(\bar{z}) + (z - \bar{z})\overline{\Phi'(z)}, \quad z \in S. \tag{4.2}$$

By periodicity of $\Phi(z)$We known that $\Omega(z)$ is also periodic.

Assume both $\Phi(z)$ and $\Omega(z)$ are at most integrably unbounded at the tips of l_k and

$$\lim_{z \to t} y\Phi'(z) = 0 (z = x + iy \in S, t \in L_0). \tag{4.3}$$

Obviously,

$$\left.\begin{aligned} X_k &= \int_{l_k} [\tau_{xy}^-(t) - \tau_{xy}^+(t)]dt, \\ Y_k &= \int_{l_k} [\sigma_y^-(t) - \sigma_y^+(t)]dt, \end{aligned}\right\} \quad k = 1, ..., n, \tag{4.4}$$

where $\sigma_y^\pm + i\sigma_{xy}^\pm$ are the external stresses on the upper bank and the lower bank of l_k respectively. Thereby, their resultant $X + iY$ is given by

$$\begin{cases} X = \int_{l_0} [\tau_{xy}^-(t) - \tau_{xy}^+(t)]dt, \\ Y = \int_{l_0} [\sigma_y^-(t) - \sigma_y^+(t)]dt, \end{cases} \tag{4.5}$$

Let $\Phi_0(z)$ and $\Psi_0(z)$ be as before, then $\Phi_0(\pm\infty i) = \Psi_0(\pm\infty i) = 0$. We can easily verify that

$$\begin{cases} \Phi(\pm\infty i) = \mp\dfrac{Y - ix}{2a\pi(k+1)} + \beta, \\ \Psi(\pm\infty i) = \mp\dfrac{(k-1)Y + i(k+1)X}{2a\pi(k+1)} - \beta + k\bar{\beta} - q; \end{cases} \tag{4.6}$$

and then, by (4.1)

$$\Omega(\pm\infty i) = \pm\frac{k(Y - iX)}{2a\pi(k+1)} + k\beta - q. \tag{4.7}$$

2 The first fundamental problem

Assume $\sigma_{xy}^{\pm}(t), \tau_{xy}^{\pm}(t) \in H$ are given, and σ_-, τ_-, h_- (consequently $\sigma_+, \tau_+, h_+, \beta, q$) are also given. Find the equilibrium.

According to the boundary condition, we assure:

$$\Phi^+(t) + \Omega^-(t) = \sigma_y^+ - i\tau_{xy}^+,$$
$$\Phi^-(t) + \Omega^+(t) = \sigma_y^- - i\tau_{xy}^-.$$

By addition and subtraction, our problem is easily transferred to the following two boundary value problems:

$$[\Phi(t) + \Omega(t)]^+ + [\Phi(t) + \Omega(t)]^- = 2p(t), \tag{4.8}$$

$$[\Phi(t) - \Omega(t)]^+ - [\Phi(t) - \Omega(t)]^- = 2q(t), \tag{4.9}$$

where we have put

$$p(t) = \frac{1}{2}[\sigma_y^+(t) + \sigma_y^-(t)] - \frac{i}{2}[\tau_{xy}^+(t) + \tau_{xy}^-(t)], \tag{4.10}$$

$$q(t) = \frac{1}{2}[\sigma_y^+(t) - \sigma_y^-(t)] - \frac{i}{2}[\tau_{xy}^+(t) - \tau_{xy}^-(t)]. \tag{4.11}$$

We have

$$\begin{cases} \Phi(z) = \dfrac{X(z)}{2a\pi i}\displaystyle\int_{l_0} \dfrac{p(t)}{X^+(t)} \cot\dfrac{t-z}{a}dt + \dfrac{1}{2a\pi i}\displaystyle\int_{l_0} q(t)\cot\dfrac{t-z}{a}dt \\ \qquad +X(z)P_n(\tan\dfrac{z}{a}) + C, \\ \Omega(z) = \dfrac{X(z)}{2a\pi i}\displaystyle\int_{l_0} \dfrac{p(t)}{X^+(t)} \cot\dfrac{t-z}{a}dt - \dfrac{1}{2a\pi i}\displaystyle\int_{l_0} q(t)\cot\dfrac{t-z}{a}dt \\ \qquad +X(z)P_n(\tan\dfrac{z}{a}) - C, \end{cases} \tag{4.12}$$

where

$$X(z) = \prod_{k=1}^{n} (\tan\frac{z}{a} - \tan\frac{a_k}{a})^{-\frac{1}{2}} (\tan\frac{z}{a} - \tan\frac{b_k}{a})^{-\frac{1}{2}} \tag{4.13}$$

The radicals in which may be arbitrarily taken as a continuous branch in the z-plane cut by the periodic cracks, for instance, that branch fulfilling

$$\lim_{z \to \frac{a\pi}{2}} \tan^n \frac{z}{a} X(z) = 1.$$

And $C_0, ..., C_m$ can be easily determined by the following equations:

$$k\int_{l_k} [\Phi^+(t) - \Omega^-(t)]dt + \int_{l_k} [\Phi^+(t) - \Omega^-(t)]dt = 0, k = 1, ..., n, \tag{4.14}$$

$$k\int_{\Lambda_t} \Phi(z)dz - \int_{\Lambda_t} \Omega(z)dz = a\pi q. \tag{4.15}$$

3) The second fundamental problem

For simplicity, we assume that there occurs only one single crack in a period, that is, $\gamma_0 : [-l, l]$.

Given the periodic displacements $u^{\pm(t)} + iv^{\pm(t)}$ respectively on the upper and the lower banks of the cracks, with $u^{\pm'}(t) + iv^{\pm'}(t) \in H(A, \mu)$, the resultant of the principal vectors of the external stresses on $\gamma_0 : X + iY$, and the stresses at $z = -\infty i$ (and hence at $z = +\infty i$), find the equilibrium.

We get

$$\begin{cases} \Phi(z) = \dfrac{1}{2a\pi ki\sqrt{R(z)}} \displaystyle\int_{-l}^{l} f(t)\sqrt{R(t)} \cot\frac{t-z}{a} dt \\ \qquad + \dfrac{1}{2a\pi ki} \displaystyle\int_{-l}^{l} g(t) \cot\frac{t-z}{a} dt + \dfrac{C_0 \tan\frac{z}{a} + C_1}{k\sqrt{R(z)}} + \beta - \dfrac{q}{2k}, \\ \Omega(z) = -\dfrac{1}{2a\pi i\sqrt{R(z)}} \displaystyle\int_{-l}^{l} f(t)\sqrt{R(t)} \cot\frac{t-z}{a} dt \\ \qquad + \dfrac{1}{2a\pi i} \displaystyle\int_{-l}^{l} f(t) \cot\frac{t-z}{a} dt - \dfrac{C_0 \tan\frac{z}{a} + C_1}{\sqrt{R(z)}}, \end{cases} \tag{4.16}$$

where

$$\begin{cases} C_0 = -\dfrac{1}{2a\pi i} \displaystyle\int_{-l}^{l} f(t)\sqrt{R(t)} dt - \dfrac{iq}{2\cos\frac{l}{a}}, \\ C_1 = -\dfrac{k(Y - iX)}{2(k+1)a\pi\cos\frac{l}{a}}. \end{cases} \tag{4.17}$$

This problem was incomplete discussion by H.F.Bueckner.

Remark The corresponding mixed problem may be studied by method similar to that used here.

4.2. *Fundamental Problems of Anisotropic Elastic Plane with Periodic Collinear Cracks*

1) General comments

Assume that, in the anisotropic infinite elastic plane, there are periodically arranged rectilinear cracks $L_j, j = 0, \pm 1, \pm 2, ...$ Lying on the $x - axis$, each of which has the length $2l(l < \frac{1}{2}a\pi)$, L_0 being the interval $[-l, l]$, as shown in Fig. 1.1 Denote $L = \sum\limits_{j=-\infty}^{\infty} L_j$ and its complement L'.

Assume that there exist periodic external loads on both sides of the cracks but no external stresses at infinity. We would study respectively the cases where the loads are symmetric or anti-symmetric on L_0.

The following discussions are made under the basic assumptions that both the stresses and displacements are periodic and the stresses at infinity are bounded.Both the stress $\Phi(z_1)$ and $\Psi(z_2)$ are periodic.

At $z = \pm\infty i$, the principal vectors $X(\pm\infty i) + iY(\pm\infty i)$ of external stresses, by assumption, are zeros:

$$X(\pm\infty i) + iY(\pm\infty i) = a\pi\left[\tau_{xy}(\pm\infty i) + i\sigma_y(\pm\infty i))\right] = 0. \tag{4.18}$$

Our discussions may be restricted in the periodic strip $|Rez| < \frac{1}{2}a\pi$.

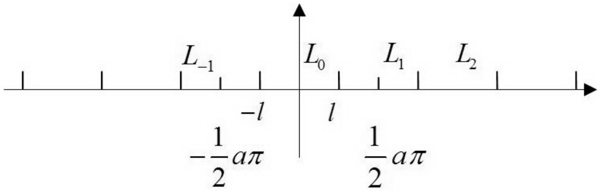

Fig. 1.1.

2) Symmetric loads

Assume $\Phi_1(z_1)$ and $\Psi_1(z_2)$ are the stress functions corresponding to the case where periodic loads (in the plane) applied on the cracks are symmetric to the x-axis, i.e., tension loads only.

On the real axis, let $z_1 = z_2 = \tau$, on account of symmetry, by(3.4)

$$\mu_1\Phi_1(\tau) + \mu_2\Psi_1(\tau) = 0, \tau \in L'. \tag{4.19}$$

Assume $\sigma_y^{\pm}(\tau), \tau \in L$, are given (periodic). By (3.5) and (4.19) the boundary condition can be expressed in $\Phi_1(z)$ only:

$$\left.\begin{aligned} \sigma_y^+(\tau) &= \frac{\mu_2 - \mu_1}{\mu_2}\Phi_1^+(\tau) + \frac{\bar{\mu}_2 - \bar{\mu}_1}{\bar{\mu}_2}\bar{\Phi}_1^-(\tau), \\ \sigma_y^-(\tau) &= \frac{\mu_2 - \mu_1}{\mu_2}\Phi_1^-(\tau) + \frac{\bar{\mu}_2 - \bar{\mu}_1}{\bar{\mu}_2}\bar{\Phi}_1^+(\tau), \end{aligned}\right\} \quad \tau \in L_0, \tag{4.20}$$

$$[\frac{\mu_2-\mu_1}{\mu_2}\Phi_1(\tau)+\frac{\bar{\mu}_2-\bar{\mu}_1}{\bar{\mu}_2}\bar{\Phi}_1(\tau)]^+ + [\frac{\mu_2-\mu_1}{\mu_2}\Phi_1(\tau)+\frac{\bar{\mu}_2-\bar{\mu}_1}{\bar{\mu}_2}\bar{\Phi}_1(\tau)]^-$$
$$= 2f_1(\tau), \tau \in L_0, \tag{4.21}$$

$$[\frac{\mu_2-\mu_1}{\mu_2}\Phi_1(\tau)-\frac{\bar{\mu}_2-\bar{\mu}_1}{\bar{\mu}_2}\bar{\Phi}_1(\tau)]^+ - [\frac{\mu_2-\mu_1}{\mu_2}\Phi_1(\tau)-\frac{\bar{\mu}_2-\bar{\mu}_1}{\bar{\mu}_2}\bar{\Phi}_1(\tau)]^-$$
$$= 2g_1(\tau), \tau \in L_0 \tag{4.22}$$

where

$$f_1(\tau) = \frac{1}{2}(\sigma_y^+ + \sigma_y^-), g_1(\tau) = \frac{1}{2}(\sigma_y^+ - \sigma_y^-), \tau \in L.$$

Assume both $f_1(\tau)$ and $g_1(\tau) \in H(A,\mu)$. Evidently, they are periodic. $\Phi_1(\pm\infty i)$ and $\Psi_1(\pm\infty i)$ are finite by the assumption that the stresses at infinity are bounded.

The general solution of the boundary value problem is

$$\Phi_1(z_1) = \Phi_1^*(z_1) + \frac{\mu_2}{\mu_2-\mu_1}\frac{c_0 \tan\frac{z_1}{a}+c_1}{\sqrt{R(z_1)}} + \frac{\mu_2\beta}{\mu_2-\mu_1}, \tag{4.23}$$

where

$$\frac{\mu_2-\mu_1}{\mu_2}\Phi_1^*(z_1) = \frac{1}{2a\pi i}\int_{-l}^{l} g_1(\tau)\cot\frac{\tau-z_1}{a}d\tau$$
$$+\frac{1}{2a\pi i\sqrt{R(z_1)}}\int_{-l}^{l} f_1(\tau)\sqrt{R(\tau)}\cot\frac{\tau-z_1}{a}d\tau. \tag{4.24}$$

By the similar way, we find that

$$\Psi_1(z_2) = \Psi_1^*(z_1) + \frac{\mu_1}{\mu_1-\mu_2}\frac{c_0^* \tan\frac{z_1}{a}+c_1^*}{\sqrt{R(z_2)}} + \frac{\mu_1\beta^*}{\mu_1-\mu_2}, \tag{4.25}$$

where

$$\frac{\mu_1-\mu_2}{\mu_1}\Psi_1^*(z_2) = \frac{1}{2a\pi i}\int_{-l}^{l} g_1(\tau)\cot\frac{\tau-z_2}{a}d\tau$$
$$+\frac{1}{2a\pi i\sqrt{R(z_2)}}\int_{-l}^{l} f_1(\tau)\sqrt{R(\tau)}\cot\frac{\tau-z_2}{a}d\tau, \tag{4.26}$$

in which c_0^*, c_1^* are undetermined real constants and β^* complex.

In order to determine $c_0, c_1, c_0^*, c_1^*, \beta$ and β^*, it is sufficient to consider the periodicity of displacements and the stresses at $z = \pm\infty i$.

3) Anti-symmetric loads

Now, let us consider the problem of anti-symmetric loads (in the plane), i.e., $\tau_{xy}^{\pm}$ are given on the cracks. denote the stress function in the case by $\Phi_2(z_1)$ and $\Psi_2(z_2)$. By the condition of anti-symmetry,

$$\Phi_2(\tau) + \Psi_2(\tau) = 0, \quad \tau \in L'.$$

Then by (3.4), the boundary conditions become

$$[(\mu_2-\mu_1)\Phi_2(\tau)+(\bar{\mu}_2-\bar{\mu}_1)\bar{\Phi}_2(\tau)]^+ + [(\mu_2-\mu_1)\Phi_2(\tau)$$

$$+(\bar{\mu}_2-\bar{\mu}_1)\bar{\Phi}_2(\tau)]^- = 2f_2(\tau), \quad \tau\in L_0. \tag{4.27}$$

$$\left[(\mu_2-\mu_1)\Phi_2(\tau)-(\bar{\mu}_2-\bar{\mu}_1)\bar{\Phi}_2(\tau)\right]^+ - [(\mu_2-\mu_1)\Phi_2(\tau)$$

$$-(\bar{\mu}_2-\bar{\mu}_1)\bar{\Phi}_2(\tau)]^- = 2g_2(\tau), \tau\in L_0 \tag{4.28}$$

where

$$f_2(\tau)=\frac{1}{2}(\tau_{xy}^+ + \tau_{xy}^-), g_2(\tau)=\frac{1}{2}(\tau_{xy}^+ - \tau_{xy}^-)$$

are periodic, assumed $\in H$.

Thus, we may write

$$\Phi_2(z_1)=\bar{\Phi}_2^*(z_1)+\frac{1}{\mu_2-\mu_1}\frac{d_0\tan\frac{z_1}{a}+d_1}{\sqrt{R(z_1)}}+\frac{\gamma}{\mu_2-\mu_1} \tag{4.29}$$

where we have

$$(\mu_2-\mu_1)\Phi_2^*(z_1)=\frac{1}{2a\pi i}\int_{-l}^{l} g_2(\tau)\cot\frac{\tau-z_1}{a}d\tau$$

$$+\frac{1}{2a\pi i\sqrt{R(z_1)}}\int_{-l}^{l} f_2(\tau)\cot\frac{\tau-z_1}{a}d\tau.$$

And

$$\Psi_2(z_2)=\Psi_2^*(z_2)+\frac{1}{\mu_1-\mu_2}\frac{d_0^*\tan\frac{z_1}{a}+d_1^*}{\sqrt{R(z_2)}}+\frac{\gamma^*}{\mu_1-\mu_2} \tag{4.30}$$

where d_0^*, d_1^* are undetermined real constants and γ^*, complex, and at the same time, we have defined

$$(\mu_1-\mu_2)\Psi_2^*(z_2)=\frac{1}{2a\pi i}\int_{-l}^{l} g_2(\tau)\cot\frac{\tau-z_2}{a}d\tau$$

$$+\frac{1}{2a\pi i\sqrt{R(z_2)}}\int_{-l}^{l} f_2(\tau)\sqrt{R(\tau)}\cot\frac{\tau-z_2}{a}d\tau,$$

$d_0^*, d_1^*\gamma, \gamma^*$ can be determined by the periodicity of displacements and the condition at $z=\pm\infty i$.

In particular, consider the subcase $\tau_{xy}^+(\tau)=\tau_{xy}^-(\tau)$ or $g_2(\tau)=0$. Then, (4.29) and (4.30) are simplified respectively to

$$(\mu_2-\mu_1)\Phi_2(z_1)=\frac{1}{2a\pi i\sqrt{R(z_1)}}\int_{-l}^{l} f_2(\tau)\sqrt{R(\tau)}\cot\frac{\tau-z_1}{a}d\tau$$

$$+\frac{d_0 \tan \frac{z_1}{a} + d_1}{\sqrt{R(z_1)}} + \gamma, \tag{4.31}$$

$$(\mu_2 - \mu_1)\Psi_2(z_2) = \frac{1}{2a\pi i\sqrt{R(z_2)}} \int_{-l}^{l} f_2(\tau)\sqrt{R(\tau)} \cot \frac{\tau - z_2}{a} d\tau$$

$$+\frac{d_0^* \tan \frac{z_1}{a} + d_1^*}{\sqrt{R(z_1)}} + \gamma^*. \tag{4.32}$$

Next, consider the more special case where uniform shearing forces are applied on the cracks: $g_2(\tau) = 0$ and $f_2(\tau) = -q$, $\tau \in L$. The solution becomes:

$$(\mu_2 - \mu_1)\Phi_2(z_1) = \frac{-q}{2a\pi i\sqrt{R(z_1)}} \int_{-l}^{l} \sqrt{R(\tau)} \cot \frac{\tau - z_1}{a} d\tau$$

$$+\frac{d_0 \tan \frac{z_1}{a} + d_1}{\sqrt{R(z_1)}} + \gamma \tag{4.33}$$

$$(\mu_1 - \mu_2)\Psi_2(z_2) = \frac{-q}{2a\pi i\sqrt{R(z_2)}} \int_{-l}^{l} \sqrt{R(\tau)} \cot \frac{\tau - z_2}{a} d\tau$$

$$+\frac{d_0^* \tan \frac{z_2}{a} + d_1^*}{\sqrt{R(z_2)}} + \gamma^* \tag{4.34}$$

4) Stress intensity factors

In order to analyze the stress distribution around the tips of L_0, put, in the neighborhood of $t = l$,

$$x = l + r\cos\theta, y = l + r\sin\theta$$

where r/l is assumed to be sufficiently small and polar coordinates r, θ represent respectively the radial distance of the point $z = x + iy$ to the tip $t = l$ of the crack and the angle of inclination of the radial ray to the crack.

Note that when $z_j = l$,

$$(\tan^2 \frac{z_j}{a} - \tan^2 \frac{l}{a})^{\frac{1}{2}} \approx \sec^2 \frac{l}{a}[2r(\cos\theta + \mu_j \sin\theta)]^{\frac{1}{2}}, \quad j = 1, 2.$$

Thus, the stress functions, either for the case of symmetric loads or anti-symmetric loads, can be written as

$$\left.\begin{aligned} \Phi_j(z_1) &= \frac{F_j}{[r(\cos\theta + \mu_j \sin\theta)]^{\frac{1}{2}}} + O(1) \\ \Psi_j(z_2) &= \frac{G_j}{[r(\cos\theta + \mu_j \sin\theta)]^{\frac{1}{2}}} + O(1) \end{aligned}\right\} \quad j = 1, 2 \tag{4.35}$$

where

$$F_1 = \frac{k_1 \mu_2 \cos \frac{l}{a}}{2\sqrt{2}(\mu_2 - \mu_1)}, F_2 = \frac{k_2 \cos \frac{l}{a}}{2\sqrt{2}(\mu_2 - \mu_1)},$$

$$G_1 = \frac{k_1 \mu_1 \cos \frac{l}{a}}{2\sqrt{2}(\mu_1 - \mu_2)}, G_2 = \frac{k_2 \cos \frac{l}{a}}{2\sqrt{2}(\mu_1 - \mu_2)} = -F_2.$$

where $k_j, j = 1, 2$, are called the stress intensity factors, and can be evaluated directly from the stress function $\Phi_j(z_1)$ or $\Psi_j(z_2)$, namely

$$\begin{cases} k_1 = 2\sqrt{2}\dfrac{\mu_2 - \mu_1}{\mu_2} \lim\limits_{z_1 \to t_0} (\tan \dfrac{z_1}{a} - \tan \dfrac{t_0}{a})^{\frac{1}{2}} \Phi_1(z_1), \\ k_2 = 2\sqrt{2(\mu_2 - \mu_1)} \lim\limits_{z_1 \to t_0} (\tan \dfrac{z_1}{a} - \tan \dfrac{t_0}{a})^{\frac{1}{2}} \Phi_2(z_1), \end{cases} \tag{4.36}$$

where $t_0 \in L_0$, or

$$\begin{cases} k_1 = 2\sqrt{2}\dfrac{\mu_1 - \mu_2}{\mu_1} \lim\limits_{z_2 \to t_0} (\tan \dfrac{z_2}{a} - \tan \dfrac{t_0}{a})^{\frac{1}{2}} \Psi_1(z_1), \\ k_2 = 2\sqrt{2(\mu_1 - \mu_2)} \lim\limits_{z_2 \to t_0} (\tan \dfrac{z_2}{a} - \tan \dfrac{t_0}{a})^{\frac{1}{2}} \Psi_2(z_2). \end{cases} \tag{4.37}$$

When $a \to \infty$, the stress intensity factors are identical to the result due to G. C. Sih and H. Liebowitz, see Ref.27, for the case only one single crack on the x-axis.

5. Generalized Symmetric Boundary Value Problems for Automorphic Functions

Various types of periodic problems often encountered in continuum mechanics are the concrete performances of some kind of generalized invariances, symmetric or conservations, which can be mathematically written in a unified way:

$$f[T(z)] \equiv f(z), \quad T \in \mathbf{G}, z \in \Sigma. \tag{5.1}$$

where $\mathbf{G}$ is a function group with Ω as its invariance domain, $f(z)$ is a meromorphic function within Ω, then $f(z)$ is a automorphic function in regard to $\mathbf{G}$. Therefore, most of the foregoing results can be generalized to corresponding boundary value problems for automorphic functions, and their conditions could be also relaxed more widely. See Ref.10-11,29.

5.1. *Relaxing Conditions*

1) Condition $\wedge(1,1)$

Let $s \in L$ be the coordinate, the arc length l, measuring from a fixed-point along L; $z = z(s), 0 \le s \le l$, be the equation of L; λ, δ and h be non-zero constants. Let $\varphi(z)$ be a continuous function with the modulus of continuity $\overset{*}{\omega}(\delta, \varphi)$ or $\omega(\delta, \varphi)$, defined separately by

$$\begin{cases} \overset{*}{\omega}(\delta, \varphi) \equiv \sup_{|s-s'| \le \delta} |\varphi(z(s)) - \varphi(z(s'))|, \\ \omega(\delta, \varphi) \equiv \sup_{|z_1 - z_2| \le \delta} |\varphi(z_1) - \varphi(z_2)|; \end{cases} \tag{5.2}$$

They are both non-decreasing continuous functions, positive valued. Moreover,

$$\overset{*}{\omega}(\delta, \varphi) \le \omega(\delta, \varphi) \le (1 + \frac{1}{\alpha}) \overset{*}{\omega}(\delta, \varphi),$$

$$\omega(\lambda\delta,\varphi) \leq (\lambda+1)(1+\frac{1}{\alpha})\omega(\delta,\varphi).$$

Corresponding to Hölder-Lipschitz Condition $\mathbf{H}(H,\alpha)$, we know

$$\omega(\delta,\varphi) \leq H\delta^{\alpha}. \tag{5.3}$$

Let $\varphi(z)$ be a function satisfied Condition $\wedge$(1,1):

$$\int_0^1 \frac{\omega(t,\phi)}{t}dt < \infty; \quad \wedge(1,0)$$

$$\int_0^1 \frac{1}{t}dt\int_0^t \frac{\omega(\tau,\phi)}{\tau}d\tau < \infty, \int_0^1 dt\int_0^t \frac{\omega(\tau,\phi)}{\tau^2}d\tau < \infty. \quad \wedge(0,1).$$

Let $\omega(\delta, \varphi)$ be substituted by $\overset{*}{\omega}(\delta, \varphi)$, accordingly we have $\overset{*}{\wedge}$(1,0), $\overset{*}{\wedge}$(0,1) and $\overset{*}{\wedge}$(1,1).

The total functions satisfied Condition $\boldsymbol{H}$(H,α), Condition $\overset{*}{\mathbf{H}}$ (H,α), Condition $\wedge$(1,1) or Condition $\overset{*}{\wedge}$(1,1), composed Group $\boldsymbol{H}$(H,α), Group $\overset{*}{\mathbf{H}}$ (H,α), Group $\wedge$(1,1) or Group $\overset{*}{\wedge}$(1,1) separately.

2) Sectionally Automorphic Functions

Let $L_k, k \equiv 0,1,2,\ldots$ be a set of rectifiable closed contours, non-intersecting to each other, oriented counter-clockwise, the region interior to L_k is denoted by S_k^+ and S^- is the complement of $S_k^+ + L_k$; for $L = \sum_{k=-\infty}^{+\infty} L_k$ correspondingly we have S^+ and S^-, as usual. Let G be a function group (Fuchs group or Elementary group) with its elements,

$\boldsymbol{T(z)\equiv z}$, $T\boldsymbol{(z)}, \boldsymbol{T(z)}, \ldots,$

where $\boldsymbol{T(z)}$ is a linear transformation, and $L_k \equiv \boldsymbol{T}(L_0)$ is a equivalence curve of L_0. $\varphi(z)$ is a sectionally automorphic function; (5.1) is valid if $\varphi(z)$ is a automorphic function for G; $\varphi(z)$ is holomorphic except the points in L and can be extended to L continuously.

5.2. *Generalized Plemelj Formulae and Singular Integral Equations*

1) Generalized Plemelj Formula

Let $\varphi(z)$ be a function satisfied Condition$\wedge$(1,1) in $\boldsymbol{L}$ and consider the Cauchy Type Integral:

$$\Phi(z) \equiv \frac{1}{2\pi i}\int_L \frac{\varphi(t)}{t-z}dt, \quad z \notin L. \tag{5.4}$$

The Generalized Plemelj Formulae can easily be proved, see Ref.11, 29.

Theorem 12 (Generalized Plemelj Formulae) Let $\varphi(z)$ be a function satisfied Condition $\wedge(1,0)$ in L, then the Cauchy Type Integral $\Phi(z)$ is holomorphic both on S^+ and S^-, vanishes at infinity; when z approaches to z_0, $\Phi(z)$ tends to definite limit $\Phi^+(z_0)$ or $\Phi^-(z_0)$ according to z in S^+ or S^-, they satisfy the following relations:

$$\begin{cases} \Phi^+(z_0) + \Phi^-(z_0) = \dfrac{1}{\pi i}\displaystyle\int_L \dfrac{\varphi(t)}{t - z_0} dt, \\ \Phi^+(z_0) - \Phi^-(z_0) = \varphi(z_0). \end{cases} \tag{5.5}$$

or

$$\begin{cases} \Phi^+(z_0) = \dfrac{1}{2\pi i}\displaystyle\int_L \dfrac{\varphi(t)}{t - z_0} dt + \dfrac{1}{2}\varphi(z_0), \\ \Phi^-(z_0) = \dfrac{1}{2\pi i}\displaystyle\int_L \dfrac{\varphi(t)}{t - z_0} dt - \dfrac{1}{2}\varphi(z_0) \end{cases} \tag{5.6}$$

where the meaning of integrals should be understand for the principal value integral.

Therefore, $\Phi(z)$ is sectionally holomorphic, and $\Phi^+(z)$, $\Phi^-(z)$ are continuous in L. When z_0 is an angular point laying in L with inner angle α, the coefficient $\pm 1/2$ of $\Phi(z)$ should be changed into $+(1 + \alpha/2)$ and $-(\alpha/2)$; when z_0 is a exterior cusp, $\alpha = 0$; when z_0 is a inner cusp, $\alpha = 2\pi$.

2) Singular Integral Equation

Because $\Phi^+(z)$ and $\Phi^-(z)$ need not to satisfy the same condition, so the above results could not be applied to singular integral equation. But we have

Theorem 13 Let $\varphi(z)$ be a function satisfied Condition $\wedge(1,1)$, then $\Phi^+(z)$ and $\Phi^-(z)$ will satisfy the same condition $\wedge(1,0)$:

$$\int_0^1 \frac{\omega(t, \Phi^+)}{t} dt < +\infty; \quad \int_0^1 \frac{\omega(t, \Phi^-)}{t} dt < +\infty. \tag{5.7}$$

By rewriting (5.6), we obtain

$$\Phi^-(z_0) = \frac{1}{2\pi i}\int_L \frac{\varphi(t) - \varphi(z_0)}{t - z_0} dt.$$

Let $z_0 + h = z(s_0 + k)$, then

$$\begin{aligned} &\Phi^-(z_0 + h) - \Phi^-(z_0) \\ &= \frac{1}{2\pi i}\int_L \Big[\frac{\varphi(t) - \varphi(z_0 + h)}{t - (z_0 + h)} - \frac{\varphi(t) - \varphi(z_0)}{t - z_0}\Big] dt \\ &\equiv \frac{1}{2\pi i}\int_{L_\varepsilon} [*] dt + \frac{1}{2\pi i}\int_{\Delta_\varepsilon} [*] dt \\ &\equiv \frac{1}{2\pi i}(I_\varepsilon + J_\varepsilon), \end{aligned}$$

where $[\cdot]$ is a simple notation of integrand; let $2|\mathrm{k}|\equiv\varepsilon$, we have

$$|J_\varepsilon| \leq \int_{s_0-\varepsilon}^{s_0+\varepsilon} \frac{\overset{*}{\omega}(|s-(s_0+k)|,\varphi)}{\alpha|s-(s_0+k)|}ds + \int_{s_0-\varepsilon}^{s_0+\varepsilon} \frac{\overset{*}{\omega}(|s-s_0|,\varphi)}{\alpha|s-s_0|}ds$$
$$\leq \frac{1}{\alpha}\int_{-s-k}^{s-k} \frac{\overset{*}{\omega}(|t|,\varphi)}{|t|}dt + \frac{1}{\alpha}\int_{-\varepsilon}^{\varepsilon} \frac{\overset{*}{\omega}(|t|,\varphi)}{|t|}dt$$
$$\leq \frac{4}{\alpha}\int_0^{4|k|} \frac{\overset{*}{\omega}(t,\varphi)}{t}dt;$$

and

$$|I_\varepsilon| \leq \int_{L_\varepsilon} \Big|\frac{\varphi(t)-\varphi(z_0+h)}{t-(z_0+h)}\Big||\frac{h}{t-z_0}|dt + \pi|\varphi(z_0+h)-\varphi(z_0)|$$
$$\leq \frac{|K|}{\alpha^2}\int_{L_\varepsilon} \frac{\overset{*}{\omega}(|s-(s_0+K)|,\varphi)}{|s-(s_0+K)||s-s_0|}ds + \pi\overset{*}{\omega}(|K|,\varphi)$$
$$\leq 2K|K|\int_{L_\varepsilon} \frac{\overset{*}{\omega}(|s-(s_0+K)|,\varphi)}{|s-(s_0+K)|^2}ds + \pi\overset{*}{\omega}(|K|,\varphi)$$
$$\leq 2K|K|\int_{|K|}^{1} \frac{\overset{*}{\omega}(t)}{t^2}dt + \pi\overset{*}{\omega}(|K|,\varphi).$$

Where h, k, K are different constants, notice the above inequalities regarding to $| I_\varepsilon |$ and $| J_\varepsilon |$, then

$$\overset{*}{\omega}(t,\Phi) \leq K_1 t\int_0^1 \frac{\overset{*}{\omega}(t)}{t^2}dt + K_2\int_0^1 \frac{\overset{*}{\omega}(t)}{t}dt + \frac{1}{2}\overset{*}{\omega}(t).$$

Using Condition $\wedge(1,0)$, we have proved the second inequality in formula (5.7); to the following Cauchy Type Integral

$$\overset{*}{\Phi}(z) \equiv \frac{1}{2\pi i}\int_{L^-} \frac{\varphi(t)}{t-z}dt, \quad z\in S^-;$$

here $\overset{*}{\Phi^-}(z) = -\Phi^+(z)$, so

$$\omega(t,\Phi^+) = \omega(t,\overset{*}{\Phi^-}).$$

Because of the above result, we have the first inequality in formula (5.7), Under the condition of $\varphi^{\pm}(z)\in\wedge(1,1)$, the following corollaries are much useful and easy to prove.

Corollary 13.1 For any given continuous function $\varphi^+(z_0)$ being the boundary value of a function $\Phi(z)$ continuous on $S^+ + L$, holomorphic on S^+, the sufficient and necessary condition is

$$\frac{1}{2\pi i}\int_L \frac{\varphi^+(t)}{t-z}dt = 0, z\in S^-. \tag{5.8}$$

Corollary 13.2 For any given continuous function $\varphi^-(z_0)$ being the boundary value of a function $\Phi(z)$ continuous on $S^- + L$ holomorphic on S^-, with a given principle part $\mathbf{r}(\mathbf{z})$ at ∞ point, the sufficient and necessary condition is

$$\frac{1}{2\pi i}\int_L \frac{\varphi^-(t)}{t-z}dt - \Gamma(z) = 0, z \in S^+. \tag{5.9}$$

5.3. *Generalized Plemelj Formula and Automorphic Functions*

1) Automorphic Functions of Finite Group

For finite group $\mathbf{G} \equiv \{\mathbf{T_0(z)} \equiv \mathbf{z}, \mathbf{T_1(z)}, \cdots, \mathbf{T_{n-1}(z)}\}$, consider the function

$$\Phi(z) \equiv \frac{1}{2\pi i}\int_{L_0}\sum_{k=0}^{n-1}\Big[\frac{1}{\tau - T_k(z)} - \frac{1}{\tau - T_k(\infty)}\Big]\varphi(\tau)d\tau. \tag{5.10}$$

where $\varphi(\tau) \in \wedge(1,1), \tau \in L_0$. Easy to know that $\varphi(\infty) = 0$, and analytic everywhere, except $L = L_0 + L_1 + \ldots + L_{n-1}$ (L is total of equivalence curves of L_k with orientational invariance, its equation is τ**-** $\boldsymbol{T}(\boldsymbol{z})$=0 or $\boldsymbol{z} = \boldsymbol{T}(\tau)$, $\tau \in \boldsymbol{L}$**,** $\boldsymbol{k}$**=**1,. . . ,2n-1). Evidently, formula (5.1) valid, namely $\Phi(\boldsymbol{z})$ is an automorphic functions of finite group $\mathbf{G}$. Taking into account the character of $\Phi(\boldsymbol{z})$ in $\boldsymbol{L}$, with z replaced by $T_k^{-1}(z)$ in formula (5.10), notice that the items under the summation varying continuously; except the ($\boldsymbol{k}$+1)th item transforming into an Cauchy Type Integral, with L as its discontinuous line. Therefore, let $z \to T_k^{-1}(z_0) \in L_k, z_0 \in L_0$, we have

$$\begin{aligned}
&\Phi^{\pm}(T_k^{-1}(z_0))\\
&= \frac{1}{\pi i}\int_{L_0}\sum_{j=0}^{n-1}[\frac{1}{\tau - T_j(T_k^{-1}(z_0))} - \frac{1}{\tau - T_j(\infty)}]\varphi(\tau)d\tau \pm \frac{1}{2}\varphi(z_0)\\
&= \frac{1}{\pi i}\int_{L_0}\sum_{j=0}^{n-1}[\frac{1}{\tau - T_k(z_0)} - \frac{1}{\tau - T_k(\infty)}]\varphi(\tau)d\tau \pm \frac{1}{2}\varphi(z_0).
\end{aligned} \tag{5.11}$$

Similar to Plemelj Formula, we have

Corollary 12.1 For an automorphic function defined by (5.10), its limit value $\Phi^{\pm}(z_0), z_0 \in L_0$ exist, and

$$\left.\begin{aligned}
&\Phi^+(T_k^{-1}(z_0)) + \Phi^{-1}(T_k^{-1}(z_0))\\
&= \tfrac{1}{\pi i}\int_{L_0}\sum_{j=0}^{n-1}[\tfrac{1}{\tau - T_k(z_0)} - \tfrac{1}{\tau - T_k(\infty)}]\varphi(\tau)d\tau,\\
&\Phi^+(T_k^{-1}(z_0)) - \Phi^{-1}(T_k^{-1}(z_0)) = \varphi(z_0), \quad k = 0, 1, \ldots, n-1.
\end{aligned}\right\} \tag{5.12}$$

2) Automorphic Functions of Infinite Group

Let $\mathbf{F(z)}$ be a simple automorphic function to infinite group $\mathbf{G} \equiv \{\mathbf{T_0(z)}, \mathbf{T_1(z)}, \cdots\}$, with a simple pole z_0 in elementary region; $g(z_0) \in \wedge(1,1)$

be given in L_0. Consider the function

$$\Phi(z) \equiv \frac{1}{2\pi i} \int_{L_0} g(\tau) \frac{F'(\tau)}{F(\tau) - F(z)} d\tau. \tag{5.13}$$

It is a sectional automorphic function, with L_0 as it's discontinuous line, vanishing at each generalized equivalent point of z_0. The integral kernel can be expressed as follows:

$$\frac{F'(\tau)}{F(\tau) - F(z)} \equiv \frac{1}{\tau - z} + \Omega(\tau, t),$$

where Ω is a function continuous in L_0. Then, we have

Corollary 12.2 For a sectional automorphic function $\Phi(z)$ defined by (5.13), it's limit value $\Phi^{\pm}(z_0)(z_0 \in L_0)$ exist, and

$$\begin{cases} \Phi^+(z_0) + \Phi^-(z_0) = \dfrac{1}{\pi i} \displaystyle\int_{L_0} g(\tau) \frac{F'(\tau)}{F(\tau) - F(z)} d\tau, \\ \Phi^+(z_0) - \Phi^-(z_0) = g(z_0). \end{cases} \tag{5.14}$$

5.4. *Singular Integral Equations and Boundary Value Problems*

In the ordinary circumstances, the exact solution of a singular integral equation can only be approached by an approximating solution. However as the equation kernel is an analytic function to the main variable, by use of analytic continuation, introducing an auxiliary (analytic) function, we can transform successfully the singular integral question to a boundary value problem,and can obtain the closed solution. The basic property of singular integral equation considered here is the automorphic behavior of the kernel.

1) Automorphic Functions of Finite Group

Under the same symbols as above,consider the following singular integral question

$$a(t)\varphi(t) + \frac{b(t)}{\pi i} \int_{L_0} \sum_{k=0}^{n-1} \Big[\frac{1}{\tau - T_k(t)} - \frac{1}{\tau - T_k(\infty)} \Big] \varphi(\tau) d\tau = f(t), \tag{5.15}$$

where $a(t), b(t), c(t)$ are given functions satisfied Condition $\wedge(1,1)$, and $a^2(t) - b^2(t) = 1$.

Substituting the function $\Phi(z)$ in (5.10) as an auxiliary function into singular integral question (5.15), using relation (5.12), we know that $\Phi(z)$ satisfies boundary condition

$$\Phi^+(T_k^{-1}(t)) = \frac{a(t) - b(t)}{a(t) + b(t)} \Phi^-(T_k^{-1}(t)) + \frac{f(t)}{a(t) + b(t)}.$$

It can be rewritten as follows

$$\Phi^+(t) = G(T_k(t))\Phi^-(t) + g(T_k(t)), k = 0, 1, \dots, n-1, \tag{5.16}$$

where

$$G(t) \equiv \frac{a(t)-b(t)}{a(t)+b(t)}, \quad g(t) \equiv \frac{f(t)}{a(t)+b(t)}. \tag{5.17}$$

This is a kind of problems called Riemann Boundary Value Problem, due to the nature of its boundary condition. But it is different from the original definition of Riemann Boundary Value Problem; the essential difference between them lies in the function $\Phi(z)$ which is a given expression of sectional automorphic function to finite group $\mathbf{G}$.

2) Automorphic Functions of Infinite Group

Under the same symbols as above,consider the singular integral question (5.15), denoting by

$$a(t)\varphi(t) + \frac{b(t)}{\pi i}\int_{L_0} \frac{F'(\tau)}{F(\tau)-F(t)}\varphi(\tau)d\tau = f(t). \tag{5.18}$$

using $\Phi(z)$ in (5.13) as its auxiliary function with substituting function $\varphi(\tau)$ for $g(\tau)$, from Corollary 12.2, the above singular integral question (5.18) can be transformed immediately to Riemann Boundary Value Problem

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t). \tag{5.19}$$

According to (5.13), function $\mathbf{F(z)}$ has a single pole at z_0, so $\Phi(\infty) = 0$.

Now, let function $\mathbf{F(z)}$ in (5.18) be a sectional automorphic function with a finite number of poles, the closed solution of generalized Riemann Boundary Value Problem (5.19) can still work out in the same way.

5.5. *Boundary Value Problems and Automorphic Functions*

1) Automorphic Functions of Finite Group

The main purpose of this paragraph is to solve Riemann Boundary Problem (5.16) for finite group $\mathbf{G}$, where L_0 is closed, so do its equivalences L_k and $L \equiv L_0 + L_1 + \cdots + L_{n-1}$. The solution of homogeneous Riemann Boundary Value Problem with the condition on L is known as canonical function $X(Z)$, is equal to the product of those canonical functions $X_k(Z)$ with various conditions $L_k, k = 1, \cdots, 2n-1$.

Under the present case, we need not to structure the canonical function $X_k(Z)$ for each L_k, by the nature of canonical function, showing obviously:

$$X_k(z) = X_0(T_k(z)). \tag{5.20}$$

And the canonical function $X_0(Z)$ for each L_0 for was easily be solved, what discovered it to come first was Hilbert, therefore, this kind of questions is called the Hilbert Problem:

$$X_0(z) = (z-t_0)^{-k} e^{\Gamma_0(t)} \quad (t_0 \in L_0), \tag{5.21}$$

where κ is the index of $\mathbf{G}(t)$ for L_0:

$$\begin{cases} k \equiv \dfrac{1}{2\pi i}\displaystyle\int_{L_0} d\ln G(\tau), \\ \Gamma_0(z) \equiv \dfrac{1}{2\pi i}\displaystyle\int_{L_0} \dfrac{\ln G(\tau)}{\tau - z} d\tau. \end{cases} \tag{5.22}$$

Therefore, the canonical function $X(Z)$ for L can be written as follows:

$$X(z) = \Pi_{k=0}^{n-1} X_0(T_k(z)) = \Pi_{k=0}^{n-1}(T_k(t) - t_0)^{-k} e^{\sum\limits_{k=0}^{n-1}\Gamma_0(T_k(z))}, \tag{5.23}$$

It never vanishes everywhere, takes the generalized equivalent points of ∞-point as poles of k-order, and

$$X(z) = X(T_k(z)), \quad k = 1, 2, \ldots, n-1. \tag{5.24}$$

To find a solution $\Phi(z)$ for the homogeneous problem,

$$\Phi^+(t) = G(T_k(t))\Phi^-(t), \quad t \in L_k, k = 1, 2, \ldots, n-1, \tag{5.25}$$

so as to meet the condition $\Phi(\infty) = 0$. Eliminated $\mathbf{G}(T_k(t))$ from equations (5.24) and (5.25), we know

$$\Phi^+(t)/X^+(t) = \Phi^-(t)/X^-(t),$$

and by the principle of analytically continuation, $\Phi(\boldsymbol{z})/\boldsymbol{X}(\boldsymbol{z})$ is meromorphic on the whole plan with generalized equivalent points of ∞-point, denoted by z_∞, as its poles of (κ-1)-order. Moreover it is automorphic due to the invariance under transforming $T_k(z) \in \mathbf{G}$ and can be expressed as one of the follows:

$$\begin{cases} \Phi(z) = X(z)P_{k-1}(F^*)/[F^*(z) - F^*(z_\infty)]^{k-1}, \\ \Phi(z) = X(z)P_{k-1}(F). \end{cases} \tag{5.26}$$

where $F(z) \equiv \sum\limits_{i=0}^{n-1} T_i(z)$ and $F(z) \equiv \sum\limits_{i=0}^{n-1}[T_i(z-a)]^{-1}$ are elemental automorphic functions, a is a constant; $\mathbf{P}(\mathbf{F})$ is an arbitrary polynomial of order not larger than $(k-1)$.

(5.26) is the expression of the κ-solutions with linear independence. When $k \leq 0$, the solution for the homogeneous problem is not exist.

Now we consider non-homogeneous problem (5.16), and change the form into

$$\Phi^+(t)/X^+(t) - \Phi^-(t)/X^-(t) = g(T_k(t))/X^+(t), \quad t \in L_k, \quad k = 0, 1, \cdots, 2n-1.$$

In this way, the problem is turned to find an automorphic function according to the discontinuities in boundary L_k or $g(t) \in \wedge(1,1)$. Easy to verify that the function

$$\Psi(z) = \frac{1}{2\pi i}\sum_{k=0}^{n-1}\int_{L_0} \frac{g(\tau)}{X^+(\tau)(\tau - T_k(z))} d\tau \tag{5.27}$$

is a sectional automorphic function satisfied the above boundary condition.

Therefore, for equation (5.16), $\boldsymbol{X}(\boldsymbol{z})$ $\Psi(\boldsymbol{z})$ is a special solution, and its general solution is

$$\Phi(z) = X(z)\{\Psi(z) + P_{k-1}(F^*)/[F^*(z) - F^*(z_\infty)]^{k-1}\}, \tag{5.28}$$

where $\boldsymbol{X}(\boldsymbol{z})$ is a canonical function determined by (5.16), and $\Psi(\boldsymbol{z})$ by (5.27).

When $k < 0$, P_{k-1} must be zero, point infinity must be the zero-point of order $(-k+1)$. In order to achieve these requirements, we expand $\Psi(\boldsymbol{z})$ in the neighborhood of infinity V(∞) and obtain

$$\Psi(z) = -\frac{1}{2\pi i}\sum_{j=1}^{\infty}\Big[\sum_{k=0}^{n-1}\int_{L_0}\frac{T_k'(\tau){T_k}^{j-1}(\tau)g(\tau)}{X^+(\tau)}d\tau\Big]z^{-j}.$$

Let all the coefficients of z^{-j} be vanished, we have the conditions:

$$\int_{L_0}\frac{g(\tau)}{X^+(\tau)}\sum_{k=0}^{n-1}T_k'(\tau)T_k^{j-1}(\tau)d\tau = 0, \quad j = 1, 2, \ldots, -k. \tag{5.29}$$

Similar to usual Riemann Boundary Value Problem, we have

Theorem 14 For homogeneous problem (5.25), there are solutions of k-linear independence, when $k > 0$; no solution, when $k \le 0$. For non-homogeneous problem (5.16), there is solution unconditionally, when $k > 0$; there is solution, only when the conditions (5.29) are satisfied.

2) Automorphic Functions of Infinite Group

Riemann Boundary Value Problem discussed here is to find a sectional automorphic function $\Phi(z)$ satisfied the boundary condition (5.19), where $\boldsymbol{G}$ ($\boldsymbol{t}$)$\neq$0 and $\boldsymbol{g}$ ($\boldsymbol{t}$) are given in L_0 and satisfied the condition $\wedge$(1,1).

Paramount considering the jump question

$$\Phi^+(t) - \Phi^-(t) = g(t), \quad t \in L. \tag{5.30}$$

The usual Cauchy Type Integration cannot satisfy the request, because it is not a automorphic function. From Corollary 12.2, we know that the function $\Phi(z)$ defined by (5.13) is uniquely the solution of (5.30); due to vanishing at z_0. The canonical function $X(z)$ of question (5.30) satisfied the boundary condition (5.13) in the homogeneous case is a sectional automorphic function with the index k of $\boldsymbol{G}$ ($\boldsymbol{t}$).

Easy to know

$$X(z) = [F(z) - F(t_0)]^{-k}e^{\Gamma(z)}, \quad t_0 \in L_0, \tag{5.31}$$

where $\boldsymbol{F}(\boldsymbol{z})$ is a simple automorphic function used in (5.30) with $\boldsymbol{z}$ as its simple pole,

$$\Gamma(z) \equiv \frac{1}{2\pi i}\int_{L_0}\ln G(\tau)\frac{F'(\tau)}{F(\tau) - F(z)}d\tau. \tag{5.32}$$

The canonical function defined such a way can be accurate to a constant.

So the general solution of (5.30) is

$$\Phi(z) = X(z)[\Psi(z) + P_k(F)], \tag{5.33}$$

where

$$\Psi(z) \equiv \frac{1}{2\pi i} \int_{L_0} \frac{g(\tau)F'(\tau)}{X^+(\tau)[F(\tau) - F(z)]} d\tau, \tag{5.34}$$

$\boldsymbol{X}(\boldsymbol{z})$ is defined by (5.31), P_k is an arbitrary polynomial of order k.

From the application's point of view, to fine a solution vanishing at point z_0 has special importance; for example in the case of solving a problem of singular integral equation. Therefore we have to consider P_{k-1} substituting for P_k; if $k \geq 0$, $P_{k-1} \equiv 0$ is essential; if $k < 0$, then we have the conditions of solvability:

$$\int_{L_0} \frac{g(\tau)}{X^+(\tau)} [F'(\tau)]^{j-1} F'(\tau) d\tau = 0, j = 1, 2, \ldots, -k. \tag{5.35}$$

Summarizing the above statements, we have

Theorem 15 (5.33) is the solution of (5.30). If an additional condition $\Phi(z_0) = 0$ must be satisfied by the solution $\Phi(z)$, P_{k-1} has to substitute for P_k; if $k \geq 0$, $P_{k-1} \equiv 0$ is essential; if $k < 0$, for the existence of solution, then a set of solvable conditions (5.35) must be satisfied.

6. Some Closed Formulae

Two kinds of singular integral equations proposed in section-4 would be solved by use of theorems 15 and 16.

1) Singular Integral Equation (5.15)

A closed solution for singular integral equation (5.15) would be found by using the solution (5.28) of (5.16), and generalized Plemelj formulae

$$\varphi(t) = \Phi^+(t) - \Phi^-(t). \tag{6.1}$$

From (6.1) and (5.28), easy to prove:

$$\begin{aligned} \varphi(t) = \frac{1}{2}[1 + 1/G(t)]g(t) + X^+(t)[1 - 1/G(t)] \\ \{\Psi(t) - \frac{1}{2}[P_{k-1}(F^*)/[F^*(t) - F^*(z_\infty)]^{k-1}]\}. \end{aligned} \tag{6.2}$$

In the above equation, let $\mathbf{X(t)}, \mathbf{\Psi(t)}$ and $\mathbf{G(t)}, \mathbf{g(t)}$ be replaced by (5.23), (5.27) and (5.20) separately, and pay attention to condition $a^2(t) - b^2(t) = 1$, we obtain

$$\begin{aligned} \varphi(t) = a(t)f(t) - b(t)Z(t) \sum_{k=0}^{n-1} \frac{1}{\pi i} \int_{L_0} f(\tau)d\tau/Z(\tau)[\tau - T_k(t)] \\ + b(t)Z(t)P_{k-1}(F^*)/[F^*(t) - F^*(z_\infty)]^{k-1}, \end{aligned} \tag{6.3}$$

where

$$Z(t) \equiv [a(t)+b(t)]X^*(t) = [a(t)-b(t)]X^-(t)$$
$$= \Pi_{k=0}^{n-1}[T_k(t)-t_0]^{-k\exp\sum_{k=0}^{n-1}\Gamma_0(T_k(t))},$$

$$\Gamma_0(t) \equiv \frac{1}{2\pi i}\int_{L_0}\ln G(\tau)d\tau/(\tau-t), \quad G(t)\equiv [a(t)-b(t)]/[a(t)+b(t)].$$

If $k \geq 0$, then let $P_{k-1}=0$; if $k<0$, for the existence of the solution, a set of solvable conditions

$$\int_{L_0}[\sum_{k=0}^{n-1}T_k'(\tau)T_k^{j-1}(\tau)]f(\tau)d\tau/Z(\tau)=0, \quad j=1,2,\ldots,-k \tag{6.4}$$

must be satisfied.

2) Singular Integral Equation (5.19)

Using the method similar to the previous section, and sectional automorphic function (5.13), applying Corollary 12.2, according to the Plemelj formula, based on the formula similar to (5.33), we can obtain the solution of Riemann Boundary Problem, namely, the solution of Singular Integral Equation (5.19):

$$\varphi(t) = a(t)f(t) - \frac{b(t)Z(t)}{\pi i}\int_{L_0}\frac{f(t)F'(\tau)d\tau}{Z(\tau)[F(\tau)-F(t)]}$$
$$+ b(t)Z(t)P_{k-1}(F), \tag{6.5}$$

where

$$Z(t) = [a(t)+b(t)]X^+(t) = [a(t)-b(t)]X^-(t)$$
$$= [F(t)-F(t_0)]^{-k}e^{\Gamma(t)}.$$

The result here is much like what stated in the previous section. If $k>0$, then let $P_{k-1}\equiv 0$; if $k\leq 0$, for the existence of the solution, a set of solvable conditions

$$\int_{L_0}\frac{f(\tau)}{Z(\tau)}[F(\tau)]^{j-1}F'(\tau)d\tau = 0, \quad j=1,2,\ldots,-k, \tag{6.6}$$

must be satisfied.

7. Some Remarks

There were although already the rich literatures in the field of automorphic function boundary value problems, singular integral equations and its applications in mechanics, but also there are many meaningful works awaiting to solve; as space is limited, we proposed certain remarks take the end of this chapter.

1) Trigonometric function, hyperbolic function, elliptical function, modular function and so on are primary automorphic functions, and suitable to characterize the phenomenon of different periodic phenomena, or some type conservation laws in nature. Many achievements stated before, under the same controlled conditions, may be generalized to the corresponding results of automorphic functions.

2) The condition $\wedge(1{,}1)$ described by modulus of continuity is more general than the Hölder-Lipschitz condition $H(H, \alpha)$. There are many results here in condition $H(H, \alpha)$, can be conditionally generalized to those on the condition $\wedge$ (1,1).

3) Using the methods here, certain type of singular integral equations can easily be solved in a closed way, most commonly with a Cauchy kernel, a logarithm kernel or an exponent kernel, including different combination of these three kind of kernels.

4) These results have widespread and important applications to the elasticity theory and the fluid mechanics. No matter in the theory or application, there are still much more works awaiting to do.

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